A ring of symmetric Hermitian modular forms of degree 2 with integral Fourier coefficients

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Abstract

We determine the structure over $\mathbb{Z}$ of a ring of symmetric Hermitian modular forms of degree 2 with integral Fourier coefficients whose weights are multiples of 4 when the base field is the Gaussian number field $\mathbb{Q}(\sqrt{-1})$. Namely, we give a set of generators consisting of 24 modular forms. As an application of our structure theorem, we give the Sturm bounds for such Hermitian modular forms of weight k with $4 \mid k$, for $p = 2$, 3. We remark that the bounds for $p \geq 5$ are already known.

1 Introduction

Let e_4 and e_6 be the normalized Eisenstein series of respective weights 4 and 6 for $\Gamma_1 := \mathrm{SL}_2(\mathbb{Z})$, and δ the Ramanujan delta function defined by $\delta = 2^{-6} \cdot 3^{-3}(e_4^3 - e_6^2)$. For the $\mathbb{Z}$-module $M_k(\Gamma_1; \mathbb{Z})$ consisting of modular forms of weight k for Γ_1 whose Fourier coefficients are in $\mathbb{Z}$, we define a ring over $\mathbb{Z}$ as

$$A(\Gamma_1; \mathbb{Z}) := \bigoplus_{k \in \mathbb{Z}} M_k(\Gamma_1; \mathbb{Z}).$$

It is a well-known classical result that all the Fourier coefficients of the modular forms e_4, e_6 and δ are integers, and they generate $A(\Gamma_1; \mathbb{Z})$. Namely we have

$$A(\Gamma_1; \mathbb{Z}) = \mathbb{Z}[e_4, e_6, \delta].$$

In the case of Siegel modular forms for the symplectic group $\Gamma_2 := \mathrm{Sp}_2(\mathbb{Z})$ of degree 2, there is a famous result of Igusa [4]. He showed such the ring over $\mathbb{Z}$ is generated by 15 modular forms. He also showed that its set of generators is minimal.

In this paper, we consider the ring of symmetric Hermitian modular forms of degree 2 with respect to $\mathbb{Q}(\sqrt{-1})$ whose Fourier coefficients are in $\mathbb{Z}$. Since it seems to be difficult to give generators of the full space of them, we restrict ourselves to the case where the weights are multiples of 4. We remark that, the ring of Siegel modular forms whose weights are multiples of 4 is generated over $\mathbb{Z}$ by 23 modular forms. This is an easy conclusion of Igusa's result.

In our case, there exists a set of generators consisting of 24 modular forms whose weights are

$$4,\ 8,\ 12,\ 12,\ 12,\ 16,\ 16,\ 20,\ 24,\ 24,\ 28,\ 28,\ 32,$$
$$36,\ 36,\ 36,\ 40,\ 40,\ 48,\ 48,\ 52,\ 60,\ 60,\ 72,\ 84.$$

The precise statement can be found in Theorem 3.7. In Subsection 3.1, we construct explicitly these generators.

As an application of this result, we can obtain the Sturm bounds for $p = 2$, 3 in the case of Hermitian modular forms whose weights are multiples of 4 (Theorem 3.9). We remark that the Sturm bounds for $p \geq 5$ are already known in [6].

2 Preliminaries

2.1 Hermitian modular forms of degree 2

We deal with the Hermitian modular forms of degree 2 only for $\boldsymbol{K} := \mathbb{Q}(\sqrt{-1})$. Let $\mathcal{O}$ be the ring of Gaussian integers, that is, $\mathcal{O} = \mathbb{Z}[\sqrt{-1}]$.

Let $\mathbb{H}_2$ be the Hermitian upper half-space of degree 2 defined as

$$\mathbb{H}_2 := \{Z \in M_2(\mathbb{C}) \mid \tfrac{1}{2i}(Z - {}^t\overline{Z}) > 0\},$$

where ${}^t\overline{Z}$ is the transposed complex conjugate of Z.

The Hermitian modular group of degree 2

$$U_2(\mathcal{O}) := \left\{M \in M_4(\mathcal{O}) \mid {}^t\overline{M} J_2 M = J_2\right\} \quad \left(J_2 := \begin{pmatrix} 0_2 & -1_2 \\ 1_2 & 0_2 \end{pmatrix}\right)$$

acts on $\mathbb{H}_2$ by the fractional transformation

$$M\langle Z\rangle := (AZ+B)(CZ+D)^{-1}, \quad Z \in \mathbb{H}_2,\ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U_2(\mathcal{O}).$$

We denote by $M_k(U_2(\mathcal{O})) = M_k^{\mathrm{Sym}}(U_2(\mathcal{O}), \det^{k/2})$ the space of the symmetric Hermitian modular forms of weight k and character $\det^{k/2}$ with respect to $U_2(\mathcal{O})$. (We deal with modular forms with character $\det^{k/2}$, but we omit the notation). Namely, the space $M_k(U_2(\mathcal{O}))$ consists of holomorphic functions $F : \mathbb{H}_2 \longrightarrow \mathbb{C}$ that satisfy

$$F\mid_k M(Z) := \det(CZ+D)^{-k} F(M\langle Z\rangle) = \det(M)^{\frac{k}{2}} \cdot F(Z),$$

for all $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U_2(\mathcal{O})$ and $F({}^tZ) = F(Z)$. Note that $\det^{k/2}$ is the trivial character if $4 \mid k$, and $M_k(U_2(\mathcal{O})) = \{0\}$ if k is odd.

The cusp forms are characterized by the condition

$$\Phi\left(F \mid_k \begin{pmatrix} {}^t\overline{U} & 0 \\ 0 & U \end{pmatrix}\right) \equiv 0 \quad \text{for all} \quad U \in \mathrm{GL}_2(\boldsymbol{K}),$$

where Φ is the Siegel Φ-operator. We denote by $S_k(U_2(\mathcal{O}))$ the subspace consisting of all cusp forms in $M_k(U_2(\mathcal{O}))$.

2.2 Fourier expansion

Since any $F \in M_k(U_2(\mathcal{O}))$ satisfies the condition

$$F(Z + B) = F(Z) \quad \text{for all} \quad B \in \mathrm{Her}_2(\mathcal{O}),$$

it has a Fourier expansion of the form

$$F(Z) = \sum_{0 \leq H \in \Lambda_2(\boldsymbol{K})} a_F(H) e^{2\pi i \mathrm{tr}(HZ)},$$

where

$$\Lambda_2(\boldsymbol{K}) := \{H = (h_{ij}) \in \mathrm{Her}_2(\boldsymbol{K}) \mid h_{ii} \in \mathbb{Z}, 2h_{ij} \in \mathcal{O}\}.$$

For simplicity, we write $H = (m, r, s, n)$ for $H = \begin{pmatrix} m & \frac{r+si}{2} \\ \frac{r-si}{2} & n \end{pmatrix} \in \Lambda_2(\boldsymbol{K})$, and $a_F(m, r, s, n)$ for $a_F\begin{pmatrix} m & \frac{r+si}{2} \\ \frac{r-si}{2} & n \end{pmatrix}$.

For a subring R of $\mathbb{C}$, we define

$$M_k(U_2(\mathcal{O}); R) := \left\{ F = \sum_{H \in \Lambda_2(\boldsymbol{K})} a_F(H) e^{2\pi i \mathrm{tr}(HZ)} \in M_k(U_2(\mathcal{O})) \;\middle|\; a_F(H) \in R \ (\forall H \in \Lambda_2(\boldsymbol{K})) \right\}.$$

We write

$$\dot{q}_{11} := \exp(2\pi i z_{11}), \quad \dot{q}_{22} := \exp(2\pi i z_{22}),$$
$$\dot{q}_{12} := \exp\left(2\pi i \frac{z_{12} - z_{21}}{-2i}\right), \quad \ddot{q}_{12} := \exp\left(2\pi i \frac{z_{12} + z_{21}}{2}\right).$$

Then for $H = (m, r, s, n) \in \Lambda_2(\boldsymbol{K})$ we have

$$e^{2\pi i \mathrm{tr}(HZ)} = \dot{q}_{11}^m \dot{q}_{12}^r \ddot{q}_{12}^s \dot{q}_{22}^n.$$

Any $F \in M_k(U_2(\mathcal{O}); R)$ can be regarded as an element of

$$R[\![\dot{\boldsymbol{q}}]\!] := R[\dot{q}_{12}^{\pm 1}, \ddot{q}_{12}^{\pm}][\![\dot{q}_{11}, \dot{q}_{22}]\!].$$

This notation is useful for calculating the Fourier expansion of Hermitian modular forms.

We consider the Hermitian Eisenstein series of degree 2 defined as

$$E_k(Z) := \sum_{M=\left(\begin{smallmatrix} * & * \\ C & D \end{smallmatrix}\right)} (\det M)^{-\frac{k}{2}} \det(CZ+D)^{-k}, \quad Z \in \mathbb{H}_2,$$

where $k > 4$ is even and $M = \begin{pmatrix} * & * \\ C & D \end{pmatrix}$ runs over a set of representatives of $\left\{ \begin{pmatrix} * & * \\ 0_2 & * \end{pmatrix} \right\} \backslash U_2(\mathcal{O})$. Then we have

$$E_k \in M_k(U_2(\mathcal{O})).$$

Moreover $E_4 \in M_4(U_2(\mathcal{O}))$ is constructed by the Maass lift ([8]). The Fourier coefficient of E_k is given by the following formula:

Theorem 2.1 (Krieg [8] (cf. Dern [2]))**.** The Fourier coefficient $a_{E_k}(H)$ of E_k is given as follows.

$$a_{E_k}(H) = \begin{cases} 1 & \text{if } H = 0_2, \\ -\dfrac{2k}{B_k} \sigma_{k-1}(\varepsilon(H)) & \text{if } \operatorname{rank}(H) = 1, \\ \dfrac{4k(k-1)}{B_k \cdot B_{k-1,\chi_{-4}}} \displaystyle\sum_{0<d|\varepsilon(H)} d^{k-1} G_{\boldsymbol{K}}(k-2, 4\det(H)/d^2) & \text{if } \operatorname{rank}(H) = 2, \end{cases}$$

where B_m is the m-th Bernoulli number, $B_{m,\chi_{-4}}$ is the m-th generalized Bernoulli number associated with the Kronecker character $\chi_{-4} = \left(\frac{-4}{*}\right)$, $\varepsilon(H) := \max\{ l \in \mathbb{N} \,|\, l^{-1}H \in \Lambda_2(\boldsymbol{K}) \}$, and

$$G_{\boldsymbol{K}}(m, N) := \frac{1}{1 + |\chi_{-4}(N)|} (\sigma_{m,\chi_{-4}}(N) - \sigma^*_{m,\chi_{-4}}(N)),$$
$$\sigma_{m,\chi_{-4}}(N) := \sum_{0<d|N} \chi_{-4}(d) d^m, \quad \sigma^*_{m,\chi_{-4}}(N) := \sum_{0<d|N} \chi_{-4}(N/d) d^m.$$

We can construct cusp forms by using the Hermitian Eisenstein series (cf. [3], Corollary 2);

$$E_{10} - E_4 E_6 \in S_{10}(U_2(\mathcal{O})),$$
$$E_{12} - \frac{441}{691} E_4^3 - \frac{250}{691} E_6^2 \in S_{12}(U_2(\mathcal{O})).$$

2.3 Siegel modular forms of degree 2

Let $M_k(\Gamma_2)$ denote the space of the Siegel modular forms of weight k $(\in \mathbb{Z})$ for the Siegel modular group $\Gamma_2 := \mathrm{Sp}_2(\mathbb{Z})$ and $S_k(\Gamma_2)$ the subspace of the cusp forms.

Any $F \in M_k(\Gamma_2)$ has a Fourier expansion of the form

$$F(Z) = \sum_{0 \leq T \in \Lambda_2} a_F(T) e^{2\pi i \mathrm{tr}(TZ)},$$

where $Z \in \mathbb{S}_2$, $\mathbb{S}_2$ is the Siegel upper half-space of degree 2 and

$$\Lambda_2 = \mathrm{Sym}_2^*(\mathbb{Z}) := \{T = (t_{ij}) \in \mathrm{Sym}_2(\mathbb{Q}) \mid t_{ii},\ 2t_{ij} \in \mathbb{Z}\ \}$$

(the lattice in $\mathrm{Sym}_2(\mathbb{R})$ of half-integral, symmetric matrices). For simplicity, we write $T = (m, r, n)$ for $T = \begin{pmatrix} m & \frac{r}{2} \\ \frac{r}{2} & n \end{pmatrix} \in \Lambda_2$, and $a_F(m, r, n)$ for $a_F \begin{pmatrix} m & \frac{r}{2} \\ \frac{r}{2} & n \end{pmatrix}$.

Taking $q_{ij} := \exp(2\pi i z_{ij})$ with $Z = (z_{ij}) \in \mathbb{H}_2$, we have for $T = (m, r, n)$

$$e^{2\pi i \mathrm{tr}(TZ)} = q_{11}^m q_{12}^r q_{22}^n.$$

For any subring $R \subset \mathbb{C}$, we adopt the notation

$$M_k(\Gamma_2; R) := \left\{ F = \sum_{T \in \Lambda_2} a_F(T) e^{2\pi i \mathrm{tr}(TZ)} \in M_k(\Gamma_2) \,\middle|\, a_F(T) \in R\ (\forall T \in \Lambda_2) \right\},$$
$$S_k(\Gamma_2; R) := M_k(\Gamma_2; R) \cap S_k(\Gamma_2).$$

Any $F \in M_k(\Gamma_2; R)$ can be regarded as an element of

$$R[\![\boldsymbol{q}]\!] := R[q_{12}^{-1}, q_{12}][\![q_{11}, q_{22}]\!].$$

The space $\mathbb{H}_2$ contains the Siegel upper half-space of degree 2

$$\mathbb{S}_2 = \mathbb{H}_2 \cap \mathrm{Sym}_2(\mathbb{C}).$$

Hence we can define the restriction map

$$R[\![\dot{\boldsymbol{q}}]\!] \longrightarrow R[\![\boldsymbol{q}]\!]$$

via the correspondence $F \mapsto F|_{\mathbb{S}_2} := F(z_{ij})|_{z_{21}=z_{12}}$ (this means $\dot{q}_{11} \mapsto q_{11}$, $\dot{q}_{22} \mapsto q_{22}$, $\dot{q}_{12} \mapsto 1$, $\ddot{q}_{12} \mapsto q_{12}$). In particular, if $F \in M_k(U_2(\mathcal{O}); R) \subset R[\![\dot{\boldsymbol{q}}]\!]$, we have $F|_{\mathbb{S}_2} \in M_k(\Gamma_2; R) \subset R[\![\boldsymbol{q}]\!]$. This fact follows from each modularity condition. The relation among the Fourier coefficients of F and $F|_{\mathbb{S}_2}$ is given by

$$a_{F|_{\mathbb{S}_2}}(m, r, n) = \sum_{\substack{s \in \mathbb{Z} \\ 4mn - (r^2 + s^2) \geq 0}} a_F(m, r, s, n). \tag{2.1}$$

2.4 Igusa's generators over $\mathbb{Z}$

Let k be an even integer with $k \geq 4$. The Siegel Eisenstein series

$$G_k(Z) := \sum_{M=\left(\begin{smallmatrix} * & * \\ C & D \end{smallmatrix}\right)} \det(CZ+D)^{-k}, \quad Z \in \mathbb{S}_2$$

defines an element of $M_k(\Gamma_2;\mathbb{Q})$. Here, $M = \begin{pmatrix} * & * \\ C & D \end{pmatrix}$ runs over a set of representatives $\left\{ \begin{pmatrix} * & * \\ 0_2 & * \end{pmatrix} \right\} \backslash \Gamma_2$. We write $X_4 := G_4$ and $X_6 := G_6$. We set

$$\begin{aligned} X_{10} :&= -\frac{43867}{2^{10}\cdot 3^5 \cdot 5^2 \cdot 7 \cdot 53}\left(G_{10} - G_4 G_6\right), \\ X_{12} :&= -\frac{131\cdot 593 \cdot 691}{2^{11}\cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 337}\left(G_{12} - \frac{441}{691}G_4^3 - \frac{250}{691}G_6^2\right). \end{aligned}$$

Then we have $X_k \in S_k(\Gamma_2;\mathbb{Z})$ $(k = 10, 12)$ and $a_{X_{10}}(1,1,1) = a_{X_{12}}(1,1,1) = 1$.

Furthermore, we define

$$\begin{aligned} Y_{12} &:= 2^{-6}\cdot 3^{-3}(X_4^3 - X_6^2) + 2^4 \cdot 3^2 X_{12}, \\ X_{16} &:= 2^{-2}\cdot 3^{-1}(X_4X_{12} - X_6X_{10}), \\ X_{18} &:= 2^{-2}\cdot 3^{-1}(X_6X_{12} - X_4^2X_{10}), \\ X_{24} &:= 2^{-3}\cdot 3^{-1}(X_{12}^2 - X_4X_{10}^2), \\ X_{28} &:= 2^{-1}\cdot 3^{-1}(X_4X_{24} - X_{10}X_{18}), \\ X_{30} &:= 2^{-1}\cdot 3^{-1}(X_6X_{24} - X_4X_{10}X_{16}), \\ X_{36} &:= 2^{-1}\cdot 3^{-2}(X_{12}X_{24} - X_{10}^2X_{16}), \\ X_{40} &:= 2^{-2}(X_4X_{36} - X_{10}X_{30}), \\ X_{42} &:= 2^{-2}\cdot 3^{-1}(X_{12}X_{30} - X_4X_{10}X_{28}), \\ X_{48} &:= 2^{-2}(X_{12}X_{36} - X_{24}^2). \end{aligned}$$

The graded ring $A^{(m)}(\Gamma_2;R)$ over R is defined by

$$A^{(m)}(\Gamma_2;\mathbb{Z}) := \bigoplus_{k\in m\mathbb{Z}} M_k(\Gamma_2;\mathbb{Z}).$$

Theorem 2.2 (Igusa [4])**.** We have $X_k \in M_k(\Gamma_2;\mathbb{Z})$ $(k = 4,\ 6,\ \cdots,\ 48)$ and $Y_{12} \in M_{12}(\Gamma_2;\mathbb{Z})$, which generate the graded ring $A^{(2)}(\Gamma_2;\mathbb{Z})$ over $\mathbb{Z}$. Moreover, the set of 14 generators is minimal.

Remark 2.3. Actually, Igusa determined the structure of the full space $A^{(1)}(\Gamma_2;\mathbb{Z})$ by using the cusp form of weight 35. However, we do not mention a detailed discussion of this remark because it is not used in this paper.

From Igusa's result, we immediately obtain the following property.

Corollary 2.4. The ring $A^{(4)}(\Gamma_2;\mathbb{Z})$ is generated over $\mathbb{Z}$ by the following 23 generators:

$$\begin{aligned}
&S_4 := X_4, \quad S_{12} := X_{12}, \quad T_{12} := Y_{12}, \quad U_{12} := X_6^2, \quad S_{16} := X_6X_{10},\\
&T_{16} := X_{16}, \quad S_{20} := X_{10}^2, \quad S_{24} := X_{24}, \quad T_{24} := X_6X_{18},\\
&S_{28} := X_{28}, \quad T_{28} := X_{10}X_{18}, \quad S_{36} := X_{36}, \quad T_{36} := X_{18}^2,\\
&U_{36} := X_6X_{30}, \quad S_{40} := X_{40}, \quad T_{40} := X_{10}X_{30}, \quad S_{48} := X_{48},\\
&T_{48} := X_{18}X_{30}, \quad S_{52} := X_{10}X_{42}, \quad S_{60} := X_{30}^2, \quad T_{60} := X_{18}X_{42},\\
&S_{72} := X_{30}X_{42}, \quad S_{84} := X_{42}^2.
\end{aligned}$$

Let p be a prime and $\mathbb{Z}_{(p)}$ the localization of $\mathbb{Z}$ at the prime ideal $(p) = p\mathbb{Z}$, namely, $\mathbb{Z}_{(p)} = \mathbb{Q} \cap \mathbb{Z}_p$.

The Sturm bounds of the Siegel modular forms of degree 2 for any primes were initially given by Poor-Yuen [10]. Subsequently, other types bounds for primes p with $p \geq 5$ and for $p = 2$, 3 were given by Choi-Choie-Kikuta [1] and Kikuta-Takemori [7], respectively.

Theorem 2.5 (Choi-Choie-Kikuta [1], Kikuta-Takemori [7] (cf. Poor-Yuen [10]))**.** Let k be a positive integer and p any prime. Let $F \in M_k(\Gamma_2;\mathbb{Z}_{(p)})$. Suppose that $a_F(m,r,n) \equiv 0 \bmod p$ for any m, r, $n \in \mathbb{Z}$ with

$$0 \leq m,\ n \leq \left[\frac{k}{10}\right]$$

and $4mn - r^2 \geq 0$. Then, we have $F \equiv 0 \bmod p$.

2.5 Hermitian modular forms over $\mathbb{Z}[1/2, 1/3]$

We set $H_4 := E_4$ and

$$\begin{aligned}
H_8 &:= -\frac{61}{2^{10}\cdot 3^2\cdot 5^2}(E_8 - H_4^2),\\
F_{10} &:= -\frac{277}{2^9\cdot 3^3\cdot 5^2\cdot 7}(E_{10} - H_4E_6),\\
H_{12} &:= -\frac{19\cdot 691\cdot 2659}{2^{11}\cdot 3^7\cdot 5^3\cdot 7^2\cdot 73}\\
&\quad\times\left(E_{12} - \frac{3^2\cdot 7^2}{691}H_4^3 - \frac{2\cdot 5^3}{691}E_6^2 + \frac{2^9\cdot 3^4\cdot 5^2\cdot 7^2\cdot 6791}{19\cdot 691\cdot 2659}H_4H_8\right).
\end{aligned}$$

The graded ring $A^{(m)}(U_2(\mathcal{O});R)$ over R is defined by

$$A^{(m)}(U_2(\mathcal{O});R) = \bigoplus_{k\in m\mathbb{Z}} M_k(U_2(\mathcal{O});R).$$

Theorem 2.6 (Dern-Krieg [3], Kikuta-Nagaoka [5])**.** All of H_4, E_6, H_8, F_{10}, and H_{12} have Fourier coefficients in $\mathbb{Z}$, and they generate the graded ring

$$A^{(2)}(U_2(\mathcal{O}); \mathbb{Z}[1/2, 1/3]).$$

Moreover, these 5 generators are algebraically independent over $\mathbb{C}$ and we have

$$H_4|_{\mathbb{S}_2} = X_4, \quad E_6|_{\mathbb{S}_2} = X_6, \quad H_8|_{\mathbb{S}_2} = 0, \quad F_{10}|_{\mathbb{S}_2} = 6X_{10}, \quad H_{12}|_{\mathbb{S}_2} = X_{12}.$$

Remark 2.7. The ring $A^{(2)}(U_2(\mathcal{O}); R)$ coincides with the ring $A^{(1)}(U_2(\mathcal{O}); R)$ of the full space of the symmetric Hermitian modular forms, because of $M_k(U_2(\mathcal{O})) = \{0\}$ for odd k.

Let p be a prime. Let ord_p be the additive valuation on $\mathbb{Q}$ normalized so that $\mathrm{ord}_p(p) = 1$. For a formal Fourier series of the form $F = \sum_H a_F(H) e^{2\pi i \mathrm{tr}(HZ)} \in \mathbb{Q}[\![\dot{\boldsymbol{q}}]\!]$, we define $v_p(F) \in \mathbb{Z}$ as

$$v_p(F) := \inf_{H \in \Lambda_2(\boldsymbol{K})} \mathrm{ord}_p(a_F(H)).$$

Then, we have the following properties.

Lemma 2.8. (1) For any $F_i = \sum a_{F_i}(H) e^{2\pi i \mathrm{tr}(HZ)}$ $(i = 1,\, 2)$ with $v_p(F_i) > -\infty$, we have

$$v_p(F_1 F_2) = v_p(F_1) + v_p(F_2).$$

(2) We have $v_p(H_8) = 0$ for any prime p.

Proof. (1) We can easily prove this property, if we define an order for two elements of $\Lambda_2(\boldsymbol{K})$ in the same way as in [6].

(2) The statement follows from the fact that $H_8 \in M_8(U_2(\mathcal{O}); \mathbb{Z})$ and $a_{H_8}(1, 1, 1, 1) = 1$. □

Lemma 2.9. Let p be any prime and $F = \sum_{m,n \geq 0} a_{m,n}(F; \dot{q}_{12}, \ddot{q}_{12}) \dot{q}_{11}^m \dot{q}_{22}^n \in \mathbb{Z}_{(p)}[\![\dot{\boldsymbol{q}}]\!]$. Let N be a positive integer. Suppose there exists $F' \in \mathbb{Z}_{(p)}[\![\dot{\boldsymbol{q}}]\!]$ such that $F \equiv H_8 F'$ mod p and $a_{m,n}(F; \dot{q}_{12}, \ddot{q}_{12}) \equiv 0$ mod p for all m, n with $0 \leq m,\ n \leq N$. Then we have $a_{m,n}(F'; \dot{q}_{12}, \ddot{q}_{12}) \equiv 0$ mod p for all m, n with $0 \leq m,\ n \leq N - 1$.

Proof. This statement can be proved in a way similar to the proof of Lemma 4.4 in Nagaoka-Takemori [9] (see also Kikuta-Takemori [7] Lemma 5.1). In fact, the Fourier expansion of H_8 is given by

$$\begin{aligned} H_8 = \dot{q}_{11}\dot{q}_{22}(4 - 2\dot{q}_{12}^{-1} - 2\dot{q}_{12} - 2\ddot{q}_{12}^{-1} \\ + \dot{q}_{12}^{-1}\ddot{q}_{12}^{-1} + \dot{q}_{12}\ddot{q}_{12}^{-1} - 2\ddot{q}_{12} + \dot{q}_{12}^{-1}\ddot{q}_{12} + \dot{q}_{12}\ddot{q}_{12}) + \cdots . \end{aligned} \tag{2.2}$$

This completes the proof of Lemma 2.9. □

We use the Sturm bounds in subsequent sections.

Theorem 2.10 (cf. Kikuta-Nagaoka [6]). Let p be a prime with $p \geq 5$ and $F \in M_k(U_2(\mathcal{O}); \mathbb{Z}_{(p)})$. Suppose that $a_F(m, r, s, n) \equiv 0 \bmod p$ for all m, r, s, $n \in \mathbb{Z}$ with

$$0 \leq m,\ n \leq \left[\frac{k}{8}\right]$$

and $4mn - (r^2 + s^2) \geq 0$. Then we have $F \equiv 0 \bmod p$.

Remark 2.11. The statement of Theorem 2 in [6] is slightly different from this statement. Therefore we modify the proof as follows.

The assumption of Theorem 2.10 and the Sturm bound in Theorem 2.5 imply that $F|_{\mathbb{S}_2} \equiv 0 \bmod p$. Theorem 2.6 yields the existence of $F' \in M_{k-8}(U_2(\mathcal{O}); \mathbb{Z}_{(p)})$ such that $F \equiv H_8 F' \bmod p$. By Lemma 2.9, such F' satisfies the same assumption of Theorem 2.10 for the weight $k - 8$. Hence we can proceed with the inductive argument on the weight k.

In general, the Sturm bounds imply the ordinary vanishing conditions.

Corollary 2.12. Let $F \in M_k(U_2(\mathcal{O}); \mathbb{Q})$. Suppose that $a_F(m, r, s, n) = 0$ for all m, r, s, $n \in \mathbb{Z}$ with

$$0 \leq m,\ n \leq \left[\frac{k}{8}\right]$$

and $4mn - (r^2 + s^2) \geq 0$. Then we have $F = 0$.

Proof. We may apply Theorem 2.10 to F for infinitely many primes p with $p \geq 5$. □

3 Structure over $\mathbb{Z}$

3.1 Construction of generators

We set

$$\begin{aligned}
I_{12} &:= 2^{-6} \cdot 3^{-3}(H_4^3 - E_6^2) + 2^4 \cdot 3^2 H_{12},\\
J_{12} &:= E_6^2,\\
H_{16} &:= 2^{-1} \cdot 3^{-1}(E_6 F_{10} - H_4^2 H_8),\\
I_{16} &:= 2^{-2} \cdot 3^{-1}(H_4 H_{12} - H_{16}),\\
H_{20} &:= 2^{-2} \cdot 3^{-2}(F_{10}^2 - H_4 H_8^2 - 2^2 \cdot 3 H_8 H_{12}),\\
H_{24} &:= 2^{-3} \cdot 3^{-1}(H_{12}^2 - H_4 H_{20}) - 2^{-1} \cdot 3^{-1} H_8 I_{16}.
\end{aligned}$$

To construct additional generators, we temporarily use the letter K.

$$\begin{aligned}
K_{14} &:= 2^{-1} \cdot 3^{-1}(H_4 F_{10} - E_6 H_8),\\
K_{18} &:= 2^{-2} \cdot 3^{-1}(E_6 H_{12} - H_4 K_{14}),\\
K_{22} &:= 2^{-1} \cdot 3^{-1}(F_{10} H_{12} - H_8 K_{14}),\\
K_{30} &:= 2^{-1} \cdot 3^{-1}(E_6 H_{24} - K_{14} I_{16}) + 3^{-1} H_8 F_{10} I_{12}.
\end{aligned}$$

From these definitions and Theorem 2.6, it is easy to see that

$$K_{14}|_{\mathbb{S}_2} = X_4X_{10}, \quad K_{18}|_{\mathbb{S}_2} = X_{18}, \quad K_{22}|_{\mathbb{S}_2} = X_{10}X_{12}, \quad K_{30}|_{\mathbb{S}_2} = X_{30}.$$

Finally we put

$$\begin{aligned}
&I_{24} := E_6K_{18}, \quad I_{28} := 2^{-1}\cdot 3^{-1}(F_{10}K_{18} - H_4H_8I_{16}),\\
&H_{28} := 2^{-1}\cdot 3^{-1}(H_4H_{24} - I_{28}) - 3^{-1}H_8^2I_{12},\\
&H_{36} := 2^{-1}\cdot 3^{-2}(H_{12}H_{24} - I_{16}H_{20}) + 7\cdot 3^{-2}H_8H_{28} + 3^{-1}H_8^3H_{12},\\
&I_{36} := K_{18}^2, \qquad J_{36} := E_6K_{30},\\
&H_{40} := 2^{-2}(H_4H_{36} - 2^{-1}\cdot 3^{-1}F_{10}K_{30}) - 5\cdot 2^{-3}\cdot 3^{-1}H_4H_8H_{28}\\
&\qquad\qquad + 2^{-2}H_8^3H_{16} + 2^{-1}H_8I_{12}H_{20},\\
&I_{40} := 2^{-1}\cdot 3^{-1}(F_{10}K_{30} - H_4H_8H_{28}),\\
&H_{48} := 2^{-2}(H_{12}H_{36} - H_{24}^2) - 2^{-3}H_8(H_{12}H_{28} + 2H_{40}\\
&\qquad\qquad\qquad + 4H_8H_{10}^2H_{12} - 2H_4H_8^2H_{20} - 2H_4H_8^3H_{12} + 4H_8I_{12}H_{20}\\
&\qquad\qquad\qquad\qquad\qquad + 2H_8^2H_{12}I_{12} - I_{16}H_{24} - 2H_8^3I_{16} + 2I_{40}),\\
&I_{48} := K_{18}K_{30},\\
&H_{52} := 2^{-1}\cdot 3^{-1}(F_{10}K_{42} - 2H_8F_{10}^2H_{12}^2 - 2^2H_8H_{12}I_{12}H_{20}\\
&\qquad\qquad - 5H_8F_{10}I_{12}K_{22} - H_8I_{16}H_{28} - H_8^3I_{12}I_{16}),\\
&H_{60} := K_{30}^2, \qquad I_{60} := K_{18}K_{42}, \qquad H_{72} := K_{30}K_{42}, \qquad H_{84} := K_{42}^2,
\end{aligned}$$

where we put

$$K_{42} := 2^{-2}\cdot 3^{-1}(H_{12}K_{30} - K_{14}H_{28}) - 2^{-1}H_8I_{12}K_{22}.$$

Note that we have $K_{42}|_{\mathbb{S}_2} = X_{42}$.

By the above definitions and from Theorem 2.6, we can easily confirm the following property.

Proposition 3.1. We have

$$H_{k_1}|_{\mathbb{S}_2} = S_{k_1}, \; I_{k_2}|_{\mathbb{S}_2} = T_{k_2} \quad \text{and} \quad J_{k_3}|_{\mathbb{S}_2} = U_{k_3}$$

for each k_1, k_2, k_3 with

$$\begin{aligned}
&k_1 \in \{4, 12, 16, 20, 24, 28, 36, 40, 48, 52, 60, 72, 84\},\\
&k_2 \in \{12, 16, 24, 28, 36, 40, 48, 60\}, \quad k_3 \in \{12, 36\}.
\end{aligned}$$

3.2 Integralities of generators

The first our purpose is to prove that all the Fourier coefficients of the modular forms constructed in the previous subsection are integers. We start by proving several lemmas.

We write $H_4 = 1 + 2^4\cdot 3S$, $E_6 = 1 + 2^3\cdot 3^2T$ with S, $T \in \mathbb{Z}[\![\dot{\boldsymbol{q}}]\!]$.

Lemma 3.2. We have $S \equiv T \bmod 2^2 \cdot 3$.

Proof. For $H \in \Lambda_2(\boldsymbol{K})$ with $\mathrm{rank}(H) = 1$, we have

$$\begin{aligned} a_{H_4}(H) &= 2^4 \cdot 3 \cdot 5 \sum_{0<d|\varepsilon(H)} d^3, \\ a_{E_6}(H) &= -2^3 \cdot 3^2 \cdot 7 \sum_{0<d|\varepsilon(H)} d^5. \end{aligned}$$

The assertion for $\mathrm{rank}(H) = 1$ follows from $5 \equiv -7 \bmod 2^2 \cdot 3$ and the application of the Euler congruence

$$\sum_{0<d|\varepsilon(H)} d^3 \equiv \sum_{0<d|\varepsilon(H)} d^5 \bmod 2^2 \cdot 3.$$

Let $H \in \Lambda_2(\boldsymbol{K})$ with $\mathrm{rank}(H) = 2$. Then we have

$$\begin{aligned} a_{H_4}(H) &= -2^6 \cdot 3 \cdot 5 \sum_{0<d|\varepsilon(H)} d^3 G_{\boldsymbol{K}}(3, 4 \det H/d^2), \\ a_{E_6}(H) &= -2^5 \cdot 3^2 \cdot 5^{-1} \cdot 7 \sum_{0<d|\varepsilon(H)} d^5 G_{\boldsymbol{K}}(5, 4 \det H/d^2). \end{aligned}$$

The Euler congruence implies that

$$\sum_{0<d|\varepsilon(H)} d^3 G_{\boldsymbol{K}}(3, 4 \det H/d^2) \equiv \sum_{0<d|\varepsilon(H)} d^5 G_{\boldsymbol{K}}(5, 4 \det H/d^2) \bmod 2^2 \cdot 3.$$

On the other hand, we have

$$2^2 \cdot 5 \equiv 2^2 \cdot 5^{-1} \cdot 7 \bmod 2^2 \cdot 3.$$

Therefore, the assertion holds. □

By this lemma, we can put $T = S + 2^2 \cdot 3U$ with $U \in \mathbb{Z}[\![\dot{\boldsymbol{q}}]\!]$. Then we have

$$\begin{aligned} H_4 &= 1 + 2^4 \cdot 3S, \\ E_6 &= 1 + 2^3 \cdot 3^2 S + 2^5 \cdot 3^3 U. \end{aligned}$$

This is an important fact for our arguments on the integralities of generators.

On the generator I_{16} For the proof of the integrality of I_{16}, we use (as in [5]) the correspondence between the Maass space and the Kohnen plus subspace given by Krieg [8]. We briefly review this correspondence.

We define the congruence subgroup of $\Gamma_1 = \mathrm{SL}_2(\mathbb{Z})$ with level N ($N \in \mathbb{N}$) as

$$\Gamma_0^{(1)}(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1 \;\middle|\; c \equiv 0 \bmod N \right\}.$$

Let $M_k(\Gamma_0^{(1)}(4), \chi_{-4}^k)$ be the space of elliptic modular forms of weight k with character χ_{-4}^k for $\Gamma_0^{(1)}(4)$. Let $\mathcal{M}_k(U_2(\mathcal{O}))$ be the Maass space consisting of all of $F \in M_k(U_2(\mathcal{O}))$ satisfying the Maass relation. For the precise definition, see [8] (p. 676).

The Hermitian modular forms version of the Kohnen plus subspace is defined as

$$\begin{aligned} &M_k^+(\Gamma_0^{(1)}(4), \chi_{-4}^k) \\ &\quad := \left\{ f = \sum_{n=0}^{\infty} a_f(n) q^n \in M_k(\Gamma_0^{(1)}(4), \chi_{-4}^k) \;\middle|\; a_f(n) = 0 \ \forall n \equiv 1 \bmod 4 \right\}. \end{aligned}$$

Krieg gave the isomorphism as the vector spaces

$$M_{k-1}^+(\Gamma_0^{(1)}(4), \chi_{-4}^{k-1}) \longrightarrow \mathcal{M}_k(U_2(\mathcal{O})).$$

Taking any

$$h = \sum_{n=0}^{\infty} a_h(n) q^n \in M_{k-1}^+(\Gamma_0^{(1)}(4), \chi_{-4}^{k-1})$$

with $q = e^{2\pi i \tau}$ and $\tau \in \mathbb{H}_1 := \{\tau = x + iy \mid y > 0\}$, we can construct a Hermitian modular form $\mathrm{Lift}(h) \in M_k(U_2(\mathcal{O}))$ using the relation among their Fourier coefficients

$$a_{\mathrm{Lift}(h)}(H) = \sum_{0<d|\varepsilon(H)} d^{k-1} \frac{1}{1 + |\chi_{-4}(4 \det H/d^2)|} a_h(4 \det H/d^2).$$

Lemma 3.3. We have $I_{16} \in M_{16}(U_2(\mathcal{O}); \mathbb{Z})$.

Proof. Let e_3 be the Eisenstein series of weight 3 for $\Gamma_0^{(1)}(4)$ with character χ_{-4} defined by

$$\begin{aligned} e_3 &= \sum_{n=0}^{\infty} a_{e_3}(n) q^n \\ &:= 1 - 4 \sum_{n=1}^{\infty} \left\{ \sum_{0<d|n} d^{k-1} \left(\chi_{-4}(d) - \chi_{-4}\left(\frac{n}{d}\right) \right) \right\} q^n. \end{aligned}$$

We remark that $a_{e_3}(n) = 0$ for all n with $n \equiv 1 \bmod 4$. In fact, for n and d with $n \equiv 1 \bmod 4$ and $d \mid n$, we have $\chi_{-4}(d) \neq 0$ and

$$\chi_{-4}(d) \left(\chi_{-4}(d) - \chi_{-4}\left(\frac{n}{d}\right) \right) = 1 - \chi_{-4}(n) = 0.$$

This means that $\chi_{-4}(d) - \chi_{-4}(n/d) = 0$ for any n and d with $n \equiv 1 \bmod 4$ and $d \mid n$.

We put

$$\begin{aligned} h_{15} &:= \delta(4\tau) e_3 \\ &= q^4 + 12q^6 + 64q^7 + 36q^8 - 128q^{10} + \cdots, \end{aligned}$$

where δ is the usual Ramanujan delta function defined in Introduction. Then we have $h_{15} \in M_{15}^{+}(\Gamma_0^{(1)}(4), \chi_{-4})$ because $a_{e_3}(n) = 0$ for all n with $n \equiv 1 \bmod 4$.

Therefore we can apply the isomorphism constructed by Krieg. Hence, there exists $\mathrm{Lift}(h_{15}) \in M_{16}(U_2(\mathcal{O}))$ such that

$$a_{\mathrm{Lift}(h_{15})}(H) = \sum_{0<d|\varepsilon(H)} \frac{d^{15}}{1+|\chi_{-4}(4\det H/d^2)|} a_{h_{15}}(4\det H/d^2).$$

From the definition of h_{15}, we can see that $h_{15} \equiv \delta(4\tau) \bmod 2$ because of $e_3 \equiv 1 \bmod 2$. Hence, we have $a_{h_{15}}(n) \equiv 0 \bmod 2$ for all n with $n \equiv 1 \bmod 2$. This implies that

$$\frac{1}{1+|\chi_{-4}(4\det H/d^2)|} a_{h_{15}}(4\det H/d^2) \in \mathbb{Z}$$

for each d. Namely $\mathrm{Lift}(h_{15}) \in M_{16}(U_2(\mathcal{O}); \mathbb{Z})$ follows.

By direct calculation, we see that

$$a_{I_{16}}(m,r,s,n) = a_{\mathrm{Lift}(h_{15})}(m,r,s,n) - 56a_{H_8^2}(m,r,s,n)$$

for all $(m,r,s,n) \in \Lambda_2(\boldsymbol{K})$ with $m,\ n \le 2 = [16/8]$. Applying Corollary 2.12, we obtain

$$I_{16} = \mathrm{Lift}(h_{15}) - 56H_8^2.$$

Since $\mathrm{Lift}(h_{15}) - 56H_8^2 \in M_{16}(U_2(\mathcal{O}); \mathbb{Z})$, we have the assertion $I_{16} \in M_{16}(U_2(\mathcal{O}); \mathbb{Z})$. □

Lemma 3.4. We have $6H_{12} - F_{10} + H_4^2 H_8 \equiv 0 \bmod 2^3 \cdot 3^2$.

Proof. By the definition of H_{16}, we have

$$2 \cdot 3H_{16} = E_6F_{10} - H_4^2H_8.$$

Hence, we can write as

$$2^3 \cdot 3^2 I_{16} = 6H_4H_{12} - E_6F_{10} + H_4^2H_8.$$

Since $I_{16} \in M_{16}(U_2(\mathcal{O}); \mathbb{Z})$, we have $6H_4H_{12} - E_6F_{10} + H_4^2H_8 \equiv 0 \bmod 2^3 \cdot 3^2$. Using the fact that $H_4 \equiv 1 \bmod 2^4 \cdot 3$, $E_6 \equiv 1 \bmod 2^3 \cdot 3^2$, we get

$$6H_{12} - F_{10} + H_4^2H_8 \equiv 0 \bmod 2^3 \cdot 3^2.$$

□

From this lemma, we can write as

$$6H_{12} - F_{10} + H_4^2H_8 = 2^3 \cdot 3^2 V$$

with $V \in \mathbb{Z}[\![\dot{\boldsymbol{q}}]\!]$. This description is another important factor for our arguments.

On the other generators First we remark that the integralities of $J_{12} = E_6^2$, $I_{24} = E_6K_{18}$, $I_{36} = K_{18}^2$, $J_{36} = E_6K_{30}$, $I_{48} = K_{18}K_{30}$, $H_{60} = K_{30}^2$, $I_{60} = K_{18}K_{42}$, $H_{72} = K_{30}K_{42}$, and $H_{84} = K_{42}^2$ follow from that of E_6, K_{18}, K_{22}, K_{30}, and K_{42}.

Lemma 3.5. We have the integralities of all the generators constructed in Subsection 3.1.

Proof. We prove this for H_{20}. By the definition of H_{20}, we can write as

$$H_{20} = 2^{-2} \cdot 3^{-2}(F_{10}^2 - 12H_{12}H_8 - H_4H_8^2).$$

If we use the descriptions

$$\begin{aligned} &F_{10} = 6H_{12} + H_4^2H_8 - 2^3 \cdot 3^2V, \\ &H_4 = 1 + 2^4 \cdot 3S, \\ &E_6 = 1 + 2^3 \cdot 3^2S + 2^5 \cdot 3^3U, \end{aligned}$$

then we have

$$\begin{aligned} H_{20} = H_{12}^2 &+ 32H_{12}H_8S + 4H_8^2S + 768H_{12}H_8S^2 + 384H_8^2S^2 \\ &+ 12288H_8^2S^3 + 147456H_8^2S^4 + 24H_{12}V + 4H_8V + 384H_8SV \\ &+ 9216H_8S^2V + 144V^2. \end{aligned}$$

This shows that $H_{20} \in \mathbb{Z}[H_{12}, H_8, S, U, V]$; therefore, $H_{20} \in M_{20}(U_2(\mathcal{O}); \mathbb{Z})$.

In the same way, we can confirm that all the generators are elements of $\mathbb{Z}[H_{12}, H_8, S, U, V]$. The integralities of all of the generators follow from this fact. □

Now we could prove the integralities of our generators:

Theorem 3.6. All the modular forms

$$\begin{aligned} &H_4,\ H_8,\ H_{12},\ I_{12},\ J_{12},\ H_{16},\ I_{16},\ H_{20},\ H_{24},\ I_{24},\ H_{28},\ I_{28}, \\ &H_{36},\ I_{36},\ J_{36},\ H_{40},\ I_{40},\ H_{48},\ I_{48},\ H_{52},\ H_{60},\ I_{60},\ H_{72},\ H_{84} \end{aligned}$$

and also

$$K_{14},\ K_{18},\ K_{22},\ K_{30},\ K_{42}$$

are elements of $\mathbb{Z}[\![\dot{\boldsymbol{q}}]\!]$.

3.3 Structure theorem

We are now in a position to prove the following main result.

Theorem 3.7. The graded ring $A^{(4)}(U_2(\mathcal{O});\mathbb{Z})$ over $\mathbb{Z}$ is generated by the following 24 modular forms:

$$H_4,\ H_8,\ H_{12},\ I_{12},\ J_{12},\ H_{16},\ I_{16},\ H_{20},\ H_{24},\ I_{24},\ H_{28},\ I_{28},$$
$$H_{36},\ I_{36},\ J_{36},\ H_{40},\ I_{40},\ H_{48},\ I_{48},\ H_{52},\ H_{60},\ I_{60},\ H_{72},\ H_{84}.$$

In other words, for any $F \in M_k(U_2(\mathcal{O});\mathbb{Z})$ with $4 \mid k$, there exists a polynomial with 24 variables having coefficients in $\mathbb{Z}$ such that $F = P(H_4, H_8, H_{12}, \cdots, H_{84})$.

Proof. We prove this by the induction on the weight.

For $k = 4$, the statement is clearly true. Let k_0 be a positive integer with $4 \mid k_0$. Suppose that the statement is true for all k with $k < k_0$. Let $F \in M_{k_0}(U_2(\mathcal{O});\mathbb{Z})$. Then there exists a polynomial P with 23 variables having coefficients in $\mathbb{Z}$ such that $F|_{\mathbb{S}_2} = P(S_4, S_{12}, T_{12}, \cdots, S_{84})$ because of Corollary 2.4. Then we have $F - P(H_4, H_{12}, I_{12}, \cdots, H_{84}) \in M_{k_0}(U_2(\mathcal{O});\mathbb{Z})$ and $(F - P(H_4, H_{12}, I_{12}, \cdots, H_{84}))|_{\mathbb{S}_2} = 0$. By the result of Dern-Krieg [3], there exists $F' \in M_{k_0-8}(U_2(\mathcal{O});\mathbb{Q})$ such that $F - P(H_4, H_{12}, I_{12}, \cdots, H_{84}) = H_8 F'$. Since all Fourier coefficients of $P(H_4, H_{12}, I_{12}, \cdots, H_{84})$ are in $\mathbb{Z}$, we have $H_8 F' \in M_k(U_2(\mathcal{O});\mathbb{Z})$. By $v_p(H_8) = 0$ for any prime p, we have $F' \in M_{k_0-8}(U_2(\mathcal{O});\mathbb{Z})$ because of Lemma 2.8. By the induction hypothesis, there exists a polynomial P' such that $F' = P'(H_4, H_8, H_{12}, \cdots, H_{84})$. Therefore we have

$$F = P(H_4, H_{12}, I_{12}, \cdots, H_{84}) + H_8 P'(H_4, H_8, H_{12} \cdots, H_{84}).$$

This completes the proof of Theorem 3.7. $\square$

Remark 3.8. To determine the structure of $A^{(2)}(U_2(\mathcal{O});\mathbb{Z})$ by our method, we need $K_{46} \in M_{46}(U_2(\mathcal{O});\mathbb{Z})$ such that $K_{46}|_{\mathbb{S}_2} = X_{10}X_{36}$. However, we predict that there does not exist such K_{46} because of the leading terms of the Fourier expansions. This is mainly why we restricted ourselves to the case in which the weights are multiples of 4. We also remark that we can construct $K'_{46} \in M_{46}(U_2(\mathcal{O});\mathbb{Z})$ such that $K'_{46}|_{\mathbb{S}_2} = 3X_{10}X_{36}$.

3.4 An Application

As an application, we have the following Sturm bounds for any k with $4 \mid k$.

Theorem 3.9. Let p be any prime and k an integer with $4 \mid k$. Suppose that $F \in M_k(U_2(\mathcal{O});\mathbb{Z}_{(p)})$ satisfies $a_F(m, r, s, n) \equiv 0 \bmod p$ for all $m,\ r,\ s,\ n \in \mathbb{Z}$ with

$$0 \leq m,\ n \leq \left[\frac{k}{8}\right]$$

and $4mn - (r^2 + s^2) \geq 0$. Then we have $F \equiv 0 \bmod p$.

For the proof, we prepare a lemma.

Lemma 3.10. Let $p = 2, 3$ and k be an integer with $4 \mid k$. Suppose that $F \in M_k(U_2(\mathcal{O}); \mathbb{Z})$ satisfies $F|_{\mathbb{S}_2} \equiv 0 \bmod p$. Then there exists $F' \in M_{k-8}(U_2(\mathcal{O}); \mathbb{Z})$ such that $F \equiv H_8 F' \bmod p$.

Proof. Since $F|_{\mathbb{S}_2} \equiv 0 \bmod p$, we have $\frac{1}{p}F|_{\mathbb{S}_2} \in M_k(\Gamma_2; \mathbb{Z})$. By Corollary 2.4, there exists an isobaric polynomial P with coefficients in $\mathbb{Z}$ such that $\frac{1}{p}F|_{\mathbb{S}_2} = P(S_4, S_{12}, \cdots, S_{84})$. If we put

$$G := P(H_4, H_{12}, \cdots, H_{84}),$$

then we have $G \in M_k(U_2(\mathcal{O}); \mathbb{Z})$ and $(F - pG)|_{\mathbb{S}_2} = 0$. By the result of Dern-Krieg [3], there exists $F' \in M_{k-8}(U_2(\mathcal{O}); \mathbb{Q})$ such that $F - pG = H_8 F'$. Since $v_p(F - pG) \geq 0$ and $v_p(H_8) = 0$ for all primes p with $p \geq 2$, it should follow that $F' \in M_{k-8}(U_2(\mathcal{O}); \mathbb{Z})$. Then we have $F \equiv H_8 F' \bmod p$.

This competes the proof of Lemma 3.10. □

Proof of Theorem 3.9. The statement for $p \geq 5$ is that of Theorem 2.10. Hence we prove the new case with $p = 2, 3$.

Taking a constant multiple cF with $c \in \mathbb{Z}_{(p)}^{\times}$, we may suppose that $F \in M_k(U_2(\mathcal{O}); \mathbb{Z})$. For $k = 4, 8$, we have the following as free $\mathbb{Z}$-modules:

$$M_4(U_2(\mathcal{O}); \mathbb{Z}) = H_4\mathbb{Z},$$
$$M_8(U_2(\mathcal{O}); \mathbb{Z}) = H_4^2\mathbb{Z} \oplus H_8\mathbb{Z}.$$

Since $H_4 \equiv 1 \bmod p$ and from the explicit form of the Fourier expansion of H_8 in (2.2), the statements for $k = 4, 8$ are trivial.

Let $k \geq 12$. From $[k/8] \geq [k/10]$ and by (2.1), we have $a_{F|_{\mathbb{S}_2}}(m, r, n) \equiv 0 \bmod p$ for all $m, n \in \mathbb{Z}$ with $m, n \leq [k/10]$. Hence we can apply the Sturm bound in Theorem 2.5 to $F|_{\mathbb{S}_2}$. Then we have $F|_{\mathbb{S}_2} \equiv 0 \bmod p$. By Lemma 3.10, there exists $F' \in M_{k-8}(U_2(\mathcal{O}); \mathbb{Z})$ such that $F \equiv H_8 F' \bmod p$. Then F' has the property that $a_{F'}(m, r, s, n) \equiv 0 \bmod p$ for any $m, n \in \mathbb{Z}$ with

$$0 \leq m,\ n \leq \left[\frac{k}{8}\right] - 1 = \left[\frac{k-8}{8}\right]$$

because of Lemma 2.9. Note here that $4 \mid k - 8$, and we can apply the above argument to F'.

If we apply this argument repeatedly, we have $F \equiv 0 \bmod p$. This completes the proof of Theorem 3.9. □

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