Structure theorem for mod p^m singular Siegel modular forms

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Abstract

We prove that all mod p^m singular forms of level N, degree $n+r$, and p-rank r with $n \geq r$ are congruent mod p^m to linear combinations of theta series attached to quadratic forms of rank r. Moreover, we prove that the levels of these theta series divide a number of the form "p-power $\times N$". Additionally, in some cases of mod p singular forms with smallest possible weight, we prove that the levels of theta series should be p.

1 Introduction

Let $F = \sum_{T\geq 0} a_F(T)e^{2\pi i \mathrm{tr}(TZ)}$ be the Fourier expansion of a Siegel modular form F of degree n, where T runs over all positive semi-definite half-integral matrices of size n. We say that F is a "singular (Siegel modular) form of rank r" if $a_F(T) = 0$ for all T with $\mathrm{rank}(T) > r$, and there exists some T with $\mathrm{rank}(T) = r$ such that $a_F(T) \neq 0$. Freitag [12] showed the following fundamental properties of singular Siegel modular forms.

- A Siegel modular form of weight k and degree n is singular if and only if $k < n/2$.
- The weight k of a singular form with singular rank r should be $k = r/2$.
- All singular modular forms of singular rank r are linear combinations of theta series attached to quadratic forms of matrix size r.

We should also mention here that Resnikoff [23] and much later Shimura [26] considered singular modular forms on more general tube domains.

In [4], we defined a notion of mod p^m singular form as follows. Let $F = \sum_{T\geq 0} a_F(T)e^{2\pi i \mathrm{tr}(TZ)}$ be a Siegel modular form of degree n with all Fourier coefficients $a_F(T)$ be p-integral. We say that F is a "mod p^m singular (Siegel modular) form of p-rank r" if $a_F(T) \equiv 0 \bmod p^m$ for all T with $\mathrm{rank}(T) > r$, and there exists some T with $\mathrm{rank}(T) = r$ such that $a_F(T) \not\equiv 0 \bmod p$. In the same paper, we proved that a congruence relation holds between the weight k and the p-rank r. More precisely we showed that $2k - r \equiv 0 \bmod (p-1)p^{m-1}$. This can be regarded as a mod p^m analogue to Freitag's result concerning weights. We also mentioned that we can construct some examples from any true singular modular forms and also from some Siegel–Eisenstein series of level 1. Another type of examples is provided by the work of Nagaoka [20] and Katsurada–Nagaoka [16], who studied p-adic limits of Siegel–Eisenstein series and showed in some cases that they are equal to certain genus theta series of singular rank. In particular, some of the Siegel–Eisenstein series studied by them are then congruent to linear combinations of singular theta series, i.e., true singular forms. We conjecture that all mod p^m singular forms are congruent mod p^m to true singular forms with some level, i.e., linear combinations of theta series with singular weights ($< \frac{n}{2}$) and some level.

In this paper, we mainly discuss this conjecture and prove it for many cases, in particular for "strongly" mod p^m singular forms: More precisely, we prove the following properties.

- All mod p^m singular forms of level N, degree $n+r$, and p-rank r with $n \geq r$ are congruent mod p^m to linear combinations of theta series attached to quadratic forms of rank r and of some level (Theorem 3.2 (1)).
- We show that the levels of these theta series divide $p^e N$ with some $e \in \mathbb{N}$ (Theorem 3.2 (2)).
- In the case where the weights are the smallest possible in some sense, then the levels of theta series should be p; this is proved for the case of mod p singular forms of p-rank 2 (Theorem 3.6).

Of course, our results can be regarded as the congruence version of Freitag's results. However, there are many new difficulties in mod p cases. For example, if a degree n theta series for a quadratic form S is congruent mod p to a level one modular form, then this does not specify the level of S. This is one of the main differences to the case over $\mathbb{C}$. We can use the filtration (weight) of the theta series as a kind of substitute, but this has rather weak properties. As new tools we use a modified version of the q-expansion principle and a degree n version of Kitaoka's transformation formula for theta series. These are shown within appendices of this paper and may be of independent interest.

We remark that we chose to consider only modular forms for groups of type $\Gamma_0(N)$ and quadratic nebentypus character. This allows us to consider congruences modulo rational primes (not modulo prime ideals in suitable algebraic number fields); also we wanted to avoid some delicate issues concerning congruences for Siegel modular forms of nonquadratic nebentypus. Instead, we show at the end of the main text of this paper how some of our results can be extended to the setting of mod p^m singular forms for arbitrary m.

This paper is organized as follows. In Section 2, we fix notation and definitions, and review known facts. In Section 3, we explain our conjectures and main results of this paper. In Section 4, we provide one of our main tools: An expansion of the rank r-part of the Fourier expansion of any modular form and its formulation for mod p^m singular forms of p-rank r. Then in Section 5 we focus on the mod p case. We first give a kind of Sturm bounds for mod p singular forms and use it to show that every strongly mod p singular modular form is represented by a linear combination of theta series for some quadratic forms. In Section 6, we try to specify the levels of the quadratic forms. We also discuss mod p singular forms with p-rank 2 whose weights are the smallest possible. In Section 7, we extend the results for the mod p case (precisely, results for general strongly mod p singular forms) proved in Section 5 and 6 to the mod p^m case by induction on m. Section 8 and 9 are appendices: We provide two tools that are needed for specification of the levels that we do in Section 6, namely a modified version of the q-expansion principle, and an extension to degree n of Kitaoka's transformation formula for theta series.

Note added in proof: In subsequent work we have applied these results to considerations about p-adic limits of Siegel–Eisenstein series [5].

2 Preliminaries

2.1 Siegel modular forms

Let n be a positive integer and $\mathbb{H}_n$ the Siegel upper half space of degree n defined as

$$\mathbb{H}_n := \{X + iY \mid X,\ Y \in \mathrm{Sym}_n(\mathbb{R}),\ Y > 0\},$$

where $\mathrm{Sym}_n(R)$ is the set of symmetric matrices of size n with components in R. We put

$$\mathrm{GSp}_n^+(\mathbb{R}) := \{g \in \mathrm{GL}_{2n}(\mathbb{R}) \mid {}^t g J_n g = n(g) J_n \text{ for some } n(g) > 0\},$$

where $J_n := \left(\begin{smallmatrix} 0_n & 1_n \\ -1_n & 0_n \end{smallmatrix}\right)$. We define the action of $\mathrm{GSp}_n^+(\mathbb{R})$ on $\mathbb{H}_n$ by $gZ = (AZ + B)(CZ + D)^{-1}$ for $Z \in \mathbb{H}_n$, $g \in \mathrm{GSp}_n^+(\mathbb{R})$. For a holomorphic function $F : \mathbb{H}_n \longrightarrow \mathbb{C}$ and a matrix $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{GSp}_n^+(\mathbb{R})$, we define the slash operator in the usual way;

$$(F|_k\, g)(Z) := n(g)^{\frac{nk}{2}} \det(CZ + D)^{-k} F(gZ).$$

We sometimes omit k and simply write $F|_k\, g$ as $F|g$.

Let $\Gamma_n := \mathrm{Sp}_n(\mathbb{Z})$ be the Siegel modular group of degree n, i.e.,

$$\Gamma_n := \left\{\gamma \in \mathrm{GL}_{2n}(\mathbb{Z}) \mid {}^t\gamma J_n \gamma = J_n\right\}.$$

Let N be a positive integer. In this paper, we deal mainly with the congruence subgroup $\Gamma_0^{(n)}(N)$ with level N of Γ_n defined as

$$\Gamma_0^{(n)}(N) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n \,\middle|\, C \equiv 0_n \bmod N \right\}.$$

We will also use the groups

$$\Gamma_1^{(n)}(N) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n \,\middle|\, C \equiv 0_n \bmod N,\ \det A \equiv \det D \equiv 1 \bmod N \right\},$$
$$\Gamma^{(n)}(N) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n \,\middle|\, B \equiv C \equiv 0_n \bmod N,\ A \equiv D \equiv 1_n \bmod N \right\},$$

where $\Gamma^{(n)}(N)$ is the so-called principal congruence subgroup of level N. We remark that

$$\Gamma^{(n)}(N) \subset \Gamma_1^{(n)}(N) \subset \Gamma_0^{(n)}(N) \subset \Gamma_n.$$

For a positive integer k and a Dirichlet character χ mod N, the space $M_k(\Gamma_0^{(n)}(N), \chi)$ of Siegel modular forms of weight k with character χ consisting of all of holomorphic functions $F : \mathbb{H}_n \to \mathbb{C}$ satisfying

$$(F|_k\, \gamma)(Z) = \chi(\det D)F(Z) \quad \text{for} \quad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^{(n)}(N).$$

If $n = 1$, the usual condition in the cusps should be added. When χ is a trivial character, we write simply $M_k(\Gamma_0^{(n)}(N))$ for $M_k(\Gamma_0^{(n)}(N), \chi)$.

Let $\Gamma \supset \Gamma^{(n)}(N)$. Similarly as above, we denote by $M_k(\Gamma)$ the space consisting of all of holomorphic functions $F : \mathbb{H}_n \to \mathbb{C}$ satisfying

$$(F|_k\, \gamma)(Z) = F(Z) \quad \text{for} \quad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma.$$

In this case also, we have to add the usual condition in the cusps when $n = 1$.

Let p be a prime p with $p \nmid N$, and ψ a Dirichlet character mod p. Let Γ denote either $\Gamma_0^{(n)}(p^m) \cap \Gamma^{(n)}(N)$ or $\Gamma_0^{(n)}(p^m) \cap \Gamma_1^{(n)}(N)$. In this paper, the notation $M_k(\Gamma, \psi)$ and $M_k(\Gamma)$ will also be used for such groups Γ, and these spaces are defined in the same way as above.

Note that, for any Dirichlet character χ mod N, we have

$$M_k(\Gamma^{(n)}(N)) \supset M_k(\Gamma_1^{(n)}(N)) \supset M_k(\Gamma_0^{(n)}(N), \chi).$$

When $F \in M_k(\Gamma)$ with $\Gamma \supset \Gamma^{(n)}(N)$, N is called the "level" of F. Sometimes Γ itself is also called the level of F.

Any $F \in M_k(\Gamma^{(n)}(N))$ has a Fourier expansion of the form

$$F(Z) = \sum_{0 \le T \in \frac{1}{N}\Lambda_n} a_F(T)\boldsymbol{e}(\mathrm{tr}(TZ)), \quad Z \in \mathbb{H}_n,$$

where $\boldsymbol{e}(x) := e^{2\pi i x}$,

$$\Lambda_n := \{T = (t_{ij}) \in \mathrm{Sym}_n(\mathbb{Q}) \mid t_{ii},\ 2t_{ij} \in \mathbb{Z}\ \}.$$

In particular, if $F \in M_k(\Gamma_1^{(n)}(N))$, the Fourier expansion of F is given in the form

$$F(Z) = \sum_{0 \le T \in \Lambda_n} a_F(T)\boldsymbol{e}(\mathrm{tr}(TZ)).$$

It is known that

$$M_k(\Gamma_1^{(n)}(N)) = \bigoplus_{\chi} M_k(\Gamma_0^{(n)}(N), \chi),$$

where χ runs over all the Dirichlet characters mod N. We remark that, if $F \in M_k(\Gamma_0^{(n)}(N), \chi)$, then we have

$$a_F(T[U]) = (\det U)^k \chi(\det U) a_F(T)$$

for each $T \in \Lambda_n$ and $U \in \mathrm{GL}_n(\mathbb{Z})$. Here we write as $T[U] := {}^tUTU$. In particular, if $\chi(-1) = (-1)^k$, we have $a_F(T[U]) = a_F(T)$ for each $T \in \Lambda_n$ and $U \in \mathrm{GL}_n(\mathbb{Z})$.

Let Φ be the Siegel Φ-operator defined by

$$\Phi(F)(Z') := \lim_{t \to \infty} F \begin{pmatrix} Z' & 0 \\ 0 & it \end{pmatrix},$$

where $F \in M_k(\Gamma_0^{(n)}(N), \chi)$, $Z' \in \mathbb{H}_{n-1}$, and $t \in \mathbb{R}$. As is well-known, we have $\Phi(F) \in M_k(\Gamma_0^{(n-1)}(N), \chi)$ and the Fourier expansion of $\Phi(F)$ is described as

$$\Phi(F)(Z') = \sum_{0 \le T \in \Lambda_{n-1}} a_F \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix} \boldsymbol{e}(\mathrm{tr}(TZ')).$$

Suppose that $F \in M_k(\Gamma_0^{(n)}(N), \chi)$ satisfies $\Phi(F) \neq 0$ (and then $F \neq 0$). Taking $\gamma \in \Gamma_0^{(n)}(N)$ as $\gamma = \left(\begin{smallmatrix} -1_n & 0_n \\ 0_n & -1_n \end{smallmatrix}\right)$, we have $F|_k\, \gamma = (-1)^{nk} F = \chi(-1)^n F$ because of the transformation law of F. Therefore we have $\chi(-1)^n = (-1)^{nk}$. On the other hand, by the same property of $\Phi(F) \neq 0$, we have $\chi(-1)^{n-1} = (-1)^{(n-1)k}$. These imply that $\chi(-1) = (-1)^k$. In this case (of $\Phi(F) \neq 0$), we have automatically $a_F(T[U]) = a_F(T)$ for each $T \in \Lambda_n$ and $U \in \mathrm{GL}_n(\mathbb{Z})$.

For a subring R of $\mathbb{C}$, let $M_k(\Gamma, \chi)_R$ (resp. $M_k(\Gamma)_R$) denote the R-module of all modular forms in $M_k(\Gamma, \chi)$ (resp. $M_k(\Gamma)$) whose Fourier coefficients are in R.

2.2 Congruences for modular forms

Let p be a prime and $\mathbb{Z}_{(p)}$ the set of p-integral rational numbers. Let F_i ($i = 1, 2$) be two formal power series of the form

$$F_i = \sum_{T \in \frac{1}{N}\Lambda_n} a_{F_i}(T)\boldsymbol{e}(\mathrm{tr}(TZ))$$

with $a_{F_i}(T) \in \mathbb{Z}_{(p)}$ for all $T \in \frac{1}{N}\Lambda_n$. We write $F_1 \equiv F_2 \bmod p^m$ if $a_{F_1}(T) \equiv a_{F_2}(T) \bmod p^m$ for all $T \in \frac{1}{N}\Lambda_n$.

Let $\widetilde{M}_k(\Gamma_0^{(n)}(N),\chi)_{p^m}$ be the set of $\widetilde{F} = \sum_T \widetilde{a_F(T)}\boldsymbol{e}(\mathrm{tr}(TZ))$ with $F \in M_k(\Gamma_0^{(n)}(N),\chi)_{\mathbb{Z}_{(p)}}$, where $\widetilde{a_F(T)} := a_F(T) \bmod p^m$. If $m = 1$, we write simply $\widetilde{M}_k(\Gamma_0^{(n)}(N),\chi)$ for $\widetilde{M}_k(\Gamma_0^{(n)}(N),\chi)_{p^m}$. Note that, $\widetilde{M}_k(\Gamma_0^{(n)}(N),\chi)$ is a vector space over $\mathbb{F}_p$.

We define the filtration weight as

$$\omega^n_{N,\chi,p^m}(F) := \min\{k \mid \widetilde{F} \in \widetilde{M}_k(\Gamma_0^{(n)}(N),\chi)_{p^m}\}.$$

We write also $\omega^n_{N,\chi}(F) := \omega^n_{N,\chi,p}(F)$.

Definition 2.1. Let $F \in M_k(\Gamma_0^{(n+r)}(N),\chi)_{\mathbb{Z}_{(p)}}$. We say that F is "mod p^m singular" if

- we have $a_F(T) \equiv 0 \bmod p^m$ for any $T \in \Lambda_{n+r}$ with $\mathrm{rank}(T) > r$,
- there exists $T \in \Lambda_{n+r}$ with $\mathrm{rank}(T) = r$ satisfying $a_F(T) \not\equiv 0 \bmod p$.

We call such r "p-rank" of F. Additionally if $n \geq r$, we say that such F is "strongly mod p^m singular".

Remark 2.2. (1) Such a condition "strongly singular" also plays a crucial role in Freitag's book [12] for the theory over $\mathbb{C}$ for arbitrary congruence subgroups.
(2) If F is nontrivial mod p^m singular, then F satisfies $\Phi(F) \neq 0$. This implies that $\chi(-1) = (-1)^k$ and $a_F(T[U]) = a_F(T)$ for each $T \in \Lambda_n$ and $U \in \mathrm{GL}_n(\mathbb{Z})$ (see Page 5). In other words, when dealing with nontrivial mod p^m singular modular forms, we may assume that $\chi(-1) = (-1)^k$.

Theorem 2.3 (Böcherer–Kikuta [4])**.** *Let n, r, k, N be positive integers and p an odd prime. Let χ be a quadratic Dirichlet character mod N with $\chi(-1) = (-1)^k$. Suppose that $F \in M_k(\Gamma_0^{(n+r)}(N),\chi)_{\mathbb{Z}_{(p)}}$ is mod p^m singular of p-rank r. Then we have $2k - r \equiv 0 \bmod (p-1)p^{m-1}$. In particular, r should be even.*

It is a classical result by Serre [24] that a congruence mod p for two elliptic modular forms f and g for level one implies a congruence of their weights mod $p-1$. We need a version for degree n including levels and quadratic nebentypus:

Proposition 2.4. *Let p be an odd prime and N a positive integer with $p \nmid N$. Let ψ and ψ' be two quadratic Dirichlet characters mod p and $F \in M_k(\Gamma_0^{(n)}(p^m) \cap \Gamma_1^{(n)}(N),\psi)$, $F' \in M_{k'}(\Gamma_0^{(n)}(p^m) \cap \Gamma_1^{(n)}(N),\psi')$ be two modular forms satisfying $F \equiv F' \bmod p$. Then $k - k' = t \cdot \frac{p-1}{2}$ holds for some $t \in \mathbb{Z}$ and we have*

$$\psi = \psi' \iff t \quad \textit{even}.$$

Proof. We want to apply the results from Böcherer–Nagaoka [8] to get the desired congruences for the weights. Note that in [8] only the case of level $\Gamma_1^{(n)}(N)$ with N coprime to p is covered. We may apply level change to F and F' to arrive at G and G' of level $\Gamma_1^{(n)}(N)$ with $F \equiv G \bmod p$ and $G' \equiv F' \bmod p$ with weights l and l'.

Let $\psi = \psi'$. Then we have $k \equiv l \bmod p-1$ and $k' \equiv l' \bmod p-1$. By [8], we have $l \equiv l' \bmod p-1$ and hence $k \equiv k' \bmod p-1$, i.e., t is even.

Let $\psi \neq \psi'$. Assume that ψ is nontrivial. Then we have $l \equiv k + \frac{p-1}{2} \bmod p-1$. Again by [8], we obtain $l \equiv l' \bmod p-1$ and hence $k + \frac{p-1}{2} \equiv k' \bmod p-1$. This implies that t has to be odd. This completes the proof. □

To formulate our results efficiently, we introduce the following (somewhat non-standard).

Definition 2.5. For an odd prime p and a positive integer N with $p \nmid N$, let χ and χ' be two quadratic Dirichlet characters mod pN. Suppose that $k - k' = t \cdot \frac{p-1}{2}$ holds for some $t \in \mathbb{Z}$. We write

$$\chi =' \chi'$$

if $\chi_N = \chi'_N$, and χ_p and χ'_p are related as in Proposition 2.4; i.e.,

$$\chi_p = \chi'_p \iff t \quad \text{even}.$$

Here χ_N and χ_p are the N-component and p-component of χ (and the same for χ'). In other words, we have

$$\chi =' \chi' \iff \chi = \chi' \left(\frac{*}{p}\right)^t,$$

where $\left(\frac{*}{p}\right)$ is the unique nontrivial quadratic character mod p.

Note that this notation depends on k, k' but it will always be clear form the context, which weights are involved.

2.3 Theta series for quadratic forms

For two matrices A, B, we write $A[B] := {}^tBAB$, whenever the product is defined. Let m be a positive integer. For S, $T \in \Lambda_m$, we write $S \sim T \bmod \mathrm{GL}_m(\mathbb{Z})$ if there exists $U \in \mathrm{GL}_m(\mathbb{Z})$ such that $S[U] = T$. We say that S and T are "$\mathrm{GL}_m(\mathbb{Z})$-equivalent" if $S \sim T \bmod \mathrm{GL}_m(\mathbb{Z})$. We denote by Λ_m^+ the set of all positive definite elements of Λ_m. We put $L := \Lambda_m$ or Λ_m^+. We write $L/\mathrm{GL}_m(\mathbb{Z})$ for $L/\sim$ the set of representatives of $\mathrm{GL}_m(\mathbb{Z})$-inequivalence classes in L.

Let m be even. For $S \in \Lambda_m^+$, we define the theta series of degree n in the usual way:

$$\theta_S^{(n)}(Z) := \sum_{X \in \mathbb{Z}^{m,n}} \boldsymbol{e}(\mathrm{tr}(S[X]Z)) \quad (Z \in \mathbb{H}_n),$$

where $\mathbb{Z}^{m,n}$ is the set of $m \times n$ matrices with integral components. We define the level of S as

$$\mathrm{level}(S) := \min\{N \in \mathbb{Z}_{\geq 1} \mid N(2S)^{-1} \in 2\Lambda_m\}.$$

Then $\theta_S^{(n)}$ defines an element of $M_{\frac{m}{2}}(\Gamma_0^{(n)}(N), \chi_S)$, where $N = \mathrm{level}(S)$, χ_S is a Dirichlet character mod N defined by

$$\chi_S(d) = \mathrm{sign}(d)^{\frac{m}{2}} \left(\frac{(-1)^{\frac{m}{2}} \det(2S)}{|d|} \right).$$

We denote by $\mathrm{cont}(S)$ the content of S defined as

$$\mathrm{cont}(S) := \max\{C \in \mathbb{Z}_{\geq 1} \mid C^{-1}S \in \Lambda_n\}.$$

For fixed $S \in \Lambda_m^+$ and $T \in \Lambda_n$, we put

$$A(S,T) := \sharp\{X \in \mathbb{Z}^{m,n} \mid S[X] = T\}.$$

Using this notation, we can write the Fourier expansion of the theta series in the form

$$\theta_S^{(n)}(Z) = \sum_{T \in \Lambda_n} A(S,T)\boldsymbol{e}(\mathrm{tr}(TZ)).$$

In the above definitions, S and T are restricted to symmetric half-integral matrices, but the same symbols (such as $\theta_S^{(n)}(Z)$ and $A(S,T)$) are used for symmetric matrices with rational components. Then $A(S,S)$ is the order of the automorphism group of S:

$$A(S,S) = \sharp\{U \in \mathrm{GL}_m(\mathbb{Z}) \mid S[U] = S\}.$$

By looking at minimal polynomials, one sees that $\mathrm{GL}_m(\mathbb{Z})$ cannot contain elements of order p if $p > m+1$. From this we obtain the very useful statement that

$$A(S,S) \not\equiv 0 \bmod p \quad \text{if} \quad p > m+1 \tag{2.1}$$

for any rational positive definite symmetric matrix S of size m.

For later use, we introduce some results on the representation numbers for binary quadratic forms:

Theorem 2.6 (Dirichlet, Weber (see Kani [15], Lemma 8, page 4))**.** *Let T, $T_i \in \Lambda_2$ $(i = 1,2)$ be primitive forms, i.e.,* $\mathrm{cont}(T) = \mathrm{cont}(T_i) = 1$*. Assume that T_1 and T_2 have a same discriminant $D = -\det(2T_i) < 0$. Then we have the following statements.*

(1) There are infinitely many primes l such that $A(T,l) > 0$.
(2) If there exists a prime l with $l \nmid D$ such that $A(T_1,l) > 0$ and $A(T_2,l) > 0$, then $T_1 \sim T_2$ mod $\mathrm{GL}_2(\mathbb{Z})$.

Theorem 2.7 (Böcherer-Nagaoka [7])**.** *Let n be a positive integer and p a prime with $p \geq 2n+3$. Let $S \in \Lambda_2^+$ be of level p. Then there exists $F \in M_{\frac{p+1}{2}}(\Gamma_n)_{\mathbb{Z}_{(p)}}$ such that $F \equiv \theta_S^{(n)}$ mod p.*

For a quadratic form $S \in \Lambda_m^+$, we put $\omega_{N,\chi,p^m}^n(S) := \omega_{N,\chi,p^m}^n(\theta_S^{(n)})$, $\omega_{N,\chi}^n(S) := \omega_{N,\chi,p}^n(\theta_S^{(n)})$.

3 Conjectures and Results

We begin by stating our conjecture in the most general situation which does not specify the weight and level.

Conjecture 3.1. *Let p be an odd prime. Any mod p^m singular form is congruent mod p^m to some true singular form.*

Our first result states that this conjecture is true in the strongly mod p^m singular case (if p is not small).

Theorem 3.2. *Let n, k, N be positive integers, and r an even integer with $n \geq r$. Let p be an odd prime with $p > r+1$ and χ a quadratic Dirichlet character mod N with $\chi(-1) = (-1)^k$. Suppose that $F \in M_k(\Gamma_0^{(n+r)}(N), \chi)_{\mathbb{Z}_{(p)}}$ is mod p^m singular of p-rank r. Then we have the following statements.*

(1) There are finitely many $S \in \Lambda_r^+/\mathrm{GL}_r(\mathbb{Z})$ such that

$$F \equiv \sum_S c_S \theta_S^{(n+r)} \bmod p^m \quad (c_S \in \mathbb{Z}_{(p)})$$

and $\widetilde{\theta_S^{(n)}} \in \widetilde{M}_k(\Gamma_0^{(n)}(N), \chi)_{p^{m-\nu}}$ (and hence $\omega^n_{N,\chi,p^{m-\nu}}(S) \leq k$). Here $\nu := \nu_p(c_S)$ and ν_p is the additive valuation on $\mathbb{Q}$ normalized so that $\nu_p(p) = 1$. Moreover, all S involved satisfy $\chi =' \chi_S$.

(2) For a suitable $e \in \mathbb{N}$, all of $S \in \Lambda_r^+$ appearing in (1) satisfy that $\mathrm{level}(S) \mid p^e N$.

Remark 3.3. (1) Actually, each c_S is described in terms of the primitive Fourier coefficient for S of F. For details, see the proof in Section 7.

(2) The statement on $\widetilde{\theta_S^{(n)}} \in \widetilde{M}_k(\Gamma_0^{(n)}(N), \chi)_{p^{m-\nu}}$ can be rephrased by $c_S\theta_S^{(n)} \in \widetilde{M}_k(\Gamma_0^{(n)}(N), \chi)_{p^m}$. Note also that $\nu = 0$ if $m = 1$.

(3) We emphasize that we do not know anything about e.

We expect that the theorem above holds in the most general case:

Conjecture 3.4. *Theorem 3.2 should hold for any odd prime (not only for $p > r+1$) and without the assumption $n \geq r$.*

The smallest weights (see Theorem 2.3) where we can expect mod p singular forms, which are not true singular forms, is of special interest. In these cases, we expect also the power e of p in the levels of theta series to be the smallest possible:

Conjecture 3.5. *Let n be a positive integer, r an even integer, and p an odd prime. We put*

$$k = k(p,r) := \begin{cases} r/2 + (p-1)/2 & \text{if} \quad r \equiv 2 \bmod 4,\ p \equiv -1 \bmod 4 \\ r/2 + p - 1 & \text{if} \quad r \equiv 0 \bmod 4. \end{cases}$$

Let N be a positive integer and χ a quadratic Dirichlet character mod N with $\chi(-1) = (-1)^k$. Suppose that $F \in M_k(\Gamma_0^{(n+r)}(N), \chi)_{\mathbb{Z}_{(p)}}$ is mod p singular of p-rank r. Then we have

$$F \equiv \sum_{\substack{S \in \Lambda_r^+/\mathrm{GL}_r(\mathbb{Z}) \\ \mathrm{level}(S) | pN}} c_S \theta_S^{(n+r)} \bmod p \quad (c_S \in \mathbb{Z}_{(p)}).$$

In the special case of $N = 1$, $r = 2$, we can prove this conjecture:

Theorem 3.6. *Let n be a positive integer and p a prime with $p \geq 2n + 7$. Suppose that $F \in M_{\frac{p+1}{2}}(\Gamma_{n+2})_{\mathbb{Z}_{(p)}}$ is mod p singular of p-rank 2. Then we have*

$$F \equiv \sum_{\substack{S \in \Lambda_2^+/\mathrm{GL}_2(\mathbb{Z}) \\ \mathrm{level}(S) = p}} c_S \theta_S^{(n+2)} \bmod p \quad (c_S \in \mathbb{Z}_{(p)}).$$

Supplement 3.7. *In the same situation as in Theorem 3.6, the degree one form $\Phi^{n+1}(F)$ is nonzero mod p.*

Remark 3.8. We remark that Theorem 3.6 and Supplement 3.7 have some potential for showing the nonvanishing of Fourier coefficients mod p in some interesting cases, e.g., a degree 3 modular form satisfying $\Phi^2(F) \equiv 0 \bmod p$ cannot be mod p singular. This implies the existence of some nonvanishing rank 3-Fourier coefficients $a_F(T)$ mod p of Klingen–Eisenstein series $F := E_k^{3,2}(h)$ attached to a cusp form h of degree 2, provided that its Fourier coefficients are in $\mathbb{Z}_{(p)}$. Note that such Fourier coefficients are quite delicate, given by some critical values of L-series attached to h and to $T \in \Lambda_3^+$, if h is a Hecke eigenform (see [2, 19] for more details on Klingen–Eisenstein series). To cover more general cases, versions of this for congruences modulo prime ideals in the field generated by the Hecke eigenvalues of h will be necessary.

4 Refinement of Freitag's expansion

In this section, inspired by Freitag [11, 12], we give a formal expansion of the "singular part" of the Fourier expansion of any modular form and apply it to mod p^m singular forms.

We fix some notation. Let $M_n(R)$ be the set of all $n \times n$ matrices whose components are in R. We put $M_n^{(r)}(\mathbb{Z}) := \{M \in M_n(\mathbb{Z}) \mid \mathrm{rank}(M) = r\}$ and $M_n^*(\mathbb{Z}) := M_n^{(n)}(\mathbb{Z})$. Similarly we write $\Lambda_n^{(r)} := \{T \in \Lambda_n \mid \mathrm{rank}(T) = r\}$ ($\Lambda_n^+ = \Lambda_n^{(n)}$). Let F be a modular form of degree $n + r$ with Fourier expansion

$$F(Z) = \sum_{T \in \Lambda_{n+r}} a(T) \boldsymbol{e}(\mathrm{tr}(TZ)).$$

We write $a(S) := a\left(\begin{smallmatrix} 0 & 0 \\ 0 & S \end{smallmatrix}\right)$ when $S \in \Lambda_r^+$. We define a subseries $F_{[r]}$ of F as

$$F_{[r]}(Z) := \sum_{T \in \Lambda_{n+r}^{(r)}} a(T)\boldsymbol{e}(\mathrm{tr}(TZ)).$$

In Böcherer–Raghavan [9] (see page 82 and 83), the notion of "primitive Fourier coefficient" was introduced; we denote it by $a^*(S)$ for S positive definite. Namely, $a^*(S)$ is defined by the formula

$$a(S) = \sum_{\substack{G \in \mathrm{GL}_r(\mathbb{Z}) \backslash M_r^*(\mathbb{Z}) \\ S[G^{-1}] \in \Lambda_r}} a^*(S[G^{-1}]).$$

We recall that by this formula, we can define a new $\mathrm{GL}_r(\mathbb{Z})$-invariant function $a^*(S)$ starting from the $\mathrm{GL}_r(\mathbb{Z})$-invariant function $T \longmapsto a(T)$ on Λ_r, using a generalization of the Moebius inversion formula in terms of Hecke operators for GL_r.

As explained in [9], this can also be written as

$$a(T) = \sum_{S \in \Lambda_r^+ / \mathrm{GL}_r(\mathbb{Z})} \frac{1}{\epsilon(S)} \sum_{\substack{W \in M_r^*(\mathbb{Z}) \\ S[W] = T}} a^*(S), \tag{4.1}$$

where $\epsilon(S) := A(S, S)$. We use this version.

Using the Fourier coefficients $a(S)$ with $S \in \Lambda_r^+$ and their modification, a slight refinement of Freitag's argument gives by formal rearrangement of the Fourier expansion our crucial identity. Note that the statements are only claiming the equality of the Fourier coefficients on both sides (ignoring questions concerning convergence!).

For two matrices A and B with the same number of rows, we denote by (A, B) the concatenation of A and B.

Lemma 4.1. *Let χ be a Dirichlet character mod N such that $\chi(-1) = (-1)^k$. Let $F \in M_k(\Gamma_0^{(n+r)}(N), \chi)$. Then we have*

$$F_{[r]}(Z) = \sum_{S \in \Lambda_r^+ / \mathrm{GL}_r(\mathbb{Z})} \frac{a^*(S)}{\epsilon(S)} \sum_{\substack{(X_1, X_2) \in \mathbb{Z}^{r,r} \times \mathbb{Z}^{r,n} \\ \mathrm{rank}(X_1, X_2) = r}} \boldsymbol{e}(\mathrm{tr}(S[(X_1, X_2)]Z)). \tag{4.2}$$

Observe that

$$\sum_{(X_1, X_2) \in \mathbb{Z}^{r,r} \times \mathbb{Z}^{r,n}} \boldsymbol{e}(\mathrm{tr}(S[(X_1, X_2)]Z)) = \sum_{X \in \mathbb{Z}^{r,n+r}} \boldsymbol{e}(\mathrm{tr}(S[X]Z)) = \theta_S^{(n+r)}(Z).$$

Hence if we can prove this lemma, then $F_{[r]}$ can be expressed by an infinite linear combination of (subseries of) theta series;

$$F_{[r]} = \sum_{S \in \Lambda_r^+ / \mathrm{GL}_r(\mathbb{Z})} \frac{a^*(S)}{\epsilon(S)} (\theta_S^{(n+r)})_{[r]}. \tag{4.3}$$

In other words, the Fourier coefficient $a(T)$ for $T \in \Lambda_r^+$ is given by

$$a(T) = \sum_{S \in \Lambda_r^+/\mathrm{GL_r}(\mathbb{Z})} \frac{a^*(S)}{\epsilon(S)} A(S,T).$$

We emphasize that this is a finite sum. This formula can also be derived directly from (4.1).

It should also be noted that $F_{[r]}$ is not a modular form since some part of the Fourier expansion is missing.

Proof of Lemma 4.1. For $X_1 \in \mathbb{Z}^{r,r}$, $X_2 \in \mathbb{Z}^{r,n}$, we can take $W \in M_r^*(\mathbb{Z})$ such that $(X_1, X_2) = W(G_1, G_2)$ and $\left(\begin{smallmatrix} * & * \\ G_1 & G_2 \end{smallmatrix}\right) \in \mathrm{GL}_{n+r}(\mathbb{Z})$. This W can be regarded as "gcd" of X_1 and X_2. We observe that such W is unique up to a factor in $\mathrm{GL}_r(\mathbb{Z})$ from the right. We switch this action of $\mathrm{GL}_r(\mathbb{Z})$ to (G_1, G_2). Then we can write the right hand side of (4.2) as

$$\sum_{S \in \Lambda_r^+/\mathrm{GL_r}(\mathbb{Z})} \frac{a^*(S)}{\epsilon(S)} \sum_{W \in M_r^*(\mathbb{Z})} \sum_{\substack{(G_1,G_2) \in \mathrm{GL}_r(\mathbb{Z}) \backslash \mathbb{Z}^{r,r} \times \mathbb{Z}^{r,n} \\ \left(\begin{smallmatrix} * & * \\ G_1 & G_2 \end{smallmatrix}\right) \in \mathrm{GL}_{n+r}(\mathbb{Z})}} \boldsymbol{e}(\mathrm{tr}(S[W][(G_1, G_2)]Z)). \tag{4.4}$$

We put $S[W] = T$ and we rewrite the summation over S as over T. Then (4.4) becomes

$$\sum_{T \in \Lambda_r^+} a(T) \sum_{\substack{(G_1,G_2) \in \mathrm{GL}_r(\mathbb{Z}) \backslash \mathbb{Z}^{r,r} \times \mathbb{Z}^{r,n} \\ \left(\begin{smallmatrix} * & * \\ G_1 & G_2 \end{smallmatrix}\right) \in \mathrm{GL}_{n+r}(\mathbb{Z})}} \boldsymbol{e}(\mathrm{tr}(T[(G_1, G_2)]Z)) \tag{4.5}$$

because of (4.1). If we put $U := \left(\begin{smallmatrix} * & * \\ G_1 & G_2 \end{smallmatrix}\right)$, then we have $T[(G_1, G_2)] = \left(\begin{smallmatrix} 0 & 0 \\ 0 & T \end{smallmatrix}\right)[U]$. Therefore we have $a(T) = a\left(\begin{smallmatrix} 0 & 0 \\ 0 & T \end{smallmatrix}\right) = a(T[(G_1, G_2)])$. Then (4.5) can be written as

$$\begin{aligned} &\sum_{T \in \Lambda_r^+} \sum_{\substack{(G_1,G_2) \in \mathrm{GL}_r(\mathbb{Z}) \backslash \mathbb{Z}^{r,r} \times \mathbb{Z}^{r,n} \\ \left(\begin{smallmatrix} * & * \\ G_1 & G_2 \end{smallmatrix}\right) \in \mathrm{GL}_{n+r}(\mathbb{Z})}} a(T[(G_1, G_2)])\boldsymbol{e}(\mathrm{tr}(T[(G_1, G_2)]Z)) \\ &= \sum_{T \in \Lambda_{n+r}^{(r)}} a(T)\boldsymbol{e}(\mathrm{tr}(TZ)) = F_{[r]}(Z). \end{aligned}$$

Here the first equality in this formula follows from the fact that, if $T \in \Lambda_r^+$ and (G_1, G_2) run as in the subscript, then $T[(G_1, G_2)]$ runs over all elements of $\Lambda_{n+r}^{(r)}$. $\square$

Let $Z_1 \in \mathbb{H}_r$, $Z_2 \in \mathbb{H}_n$. Consider the restriction of F to $Z = \left(\begin{smallmatrix} Z_1 & 0 \\ 0 & Z_2 \end{smallmatrix}\right)$;

$$F \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix} = \sum_{T \in \Lambda_r} \phi_T(Z_2)\mathbf{e}(\mathrm{tr}(TZ_1)).$$

Then we have $\phi_T(Z_2) \in M_k(\Gamma_0^{(n)}(N),\chi)$ for any $T \in \Lambda_r$. For the proof of this fact, we refer to Andrianov [1] (page 83 and 84).

On the other hand, for a function $G(Z_1,Z_2)$ ($Z_1 \in \mathbb{H}_r$, $Z_2 \in \mathbb{H}_n$) of the form

$$G(Z_1,Z_2) = \sum_{T\in\Lambda_r} \varphi_T(Z_2)\mathbf{e}(\mathrm{tr}(TZ_1)),$$

we put

$$G(Z_1,Z_2)^\sharp := \sum_{T\in\Lambda_r^+} \varphi_T(Z_2)\mathbf{e}(\mathrm{tr}(TZ_1)).$$

Then we have

$$F\begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix}^\sharp = \sum_{T\in\Lambda_r^+} \phi_T(Z_2)\mathbf{e}(\mathrm{tr}(TZ_1)).$$

Note that still we have $\phi_T(Z_2) \in M_k(\Gamma_0^{(n)}(N),\chi)$ for any $T \in \Lambda_r^+$.

Now assume that F is mod p^m singular of p-rank r. Then from F being mod p^m singular we obtain $F\left(\begin{smallmatrix} Z_1 & 0 \\ 0 & Z_2 \end{smallmatrix}\right) \equiv F_{[r]}\left(\begin{smallmatrix} Z_1 & 0 \\ 0 & Z_2 \end{smallmatrix}\right)$ mod p^m and hence $F\left(\begin{smallmatrix} Z_1 & 0 \\ 0 & Z_2 \end{smallmatrix}\right)^\sharp \equiv F_{[r]}\left(\begin{smallmatrix} Z_1 & 0 \\ 0 & Z_2 \end{smallmatrix}\right)^\sharp$ mod p^m. It follows from Lemma 4.1 that

$$\begin{aligned}
F_{[r]}\begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix}^\sharp &= \left(\sum_{S\in\Lambda_r^+/\mathrm{GL_r}(\mathbb{Z})} \frac{a^*(S)}{\epsilon(S)} \sum_{\substack{(X_1,X_2)\in\mathbb{Z}^{r,r}\times\mathbb{Z}^{r,n}\\ \mathrm{rank}(X_1,X_2)=r}} \boldsymbol{e}(\mathrm{tr}(S[X_1]Z_1)\boldsymbol{e}(\mathrm{tr}(S[X_2]Z_2)) \right)^\sharp \\
&= \sum_{S\in\Lambda_r^+/\mathrm{GL_r}(\mathbb{Z})} \frac{a^*(S)}{\epsilon(S)} \sum_{\substack{(X_1,X_2)\in\mathbb{Z}^{r,r}\times\mathbb{Z}^{r,n}\\ \mathrm{rank}(X_1)=r}} \boldsymbol{e}(\mathrm{tr}(S[X_1]Z_1)\boldsymbol{e}(\mathrm{tr}(S[X_2]Z_2)) \\
&= \sum_{S\in\Lambda_r^+/\mathrm{GL_r}(\mathbb{Z})} \frac{a^*(S)}{\epsilon(S)} \sum_{\substack{X_1\in\mathbb{Z}^{r,r}\\ \mathrm{rank}(X_1)=r}} \boldsymbol{e}(\mathrm{tr}(S[X_1]Z_1)) \sum_{X_2\in\mathbb{Z}^{r,n}} \boldsymbol{e}(\mathrm{tr}(S[X_2]Z_2)) \\
&= \sum_{S\in\Lambda_r^+/\mathrm{GL_r}(\mathbb{Z})} \frac{a^*(S)}{\epsilon(S)} (\theta_S^{(r)}(Z_1))_{[r]}\theta_S^{(n)}(Z_2).
\end{aligned}$$

This implies that

$$\phi_T(Z_2) \equiv \sum_{S\in\Lambda_r^+/\mathrm{GL_r}(\mathbb{Z})} A(S,T)\frac{a^*(S)}{\epsilon(S)}\theta_S^{(n)}(Z_2) \bmod p^m$$

for any $T \in \Lambda_r^+$. Hence we obtain

$$\sum_{S\in\Lambda_r^+/\mathrm{GL_r}(\mathbb{Z})} A(S,T)\frac{a^*(S)}{\epsilon(S)}\theta_S^{(n)}(Z_2) \bmod p^m \in \widetilde{M}_k(\Gamma_0^{(n)}(N),\chi)_{p^m}.$$

In the ordinary case (over $\mathbb{C}$), the key in Freitag's setting would be that $a(S)$ can be different from zero only if level$(S) \mid N$. In our mod p^m setting, we get a condition on the filtration of $\theta_S^{(n)}$ for $S \in \Lambda_r^+$ with $a^*(S) \not\equiv 0 \bmod p^m$. More precisely, we get the following property.

Proposition 4.2. *Let n, k, N be positive integers and r an even integer. Let p be an odd prime with $p > r+1$ and χ a quadratic Dirichlet character mod N with $\chi(-1) = (-1)^k$. Suppose that $F \in M_k(\Gamma_0^{(n+r)}(N), \chi)_{\mathbb{Z}_{(p)}}$ is mod p^m singular of p-rank r. Then we have $a^*(S)\theta_S^{(n)}$ mod $p^m \in \widetilde{M}_k(\Gamma_0^{(n)}(N), \chi)_{p^m}$ for any $S \in \Lambda_r^+$. In particular, if $S \in \Lambda_r^+$ satisfies $a^*(S) \not\equiv 0$ mod p^m, then we have $\theta_S^{(n)}$ mod $p^{m-\nu}$ $\in \widetilde{M}_k(\Gamma_0^{(n)}(N), \chi)_{p^{m-\nu}}$ (and hence $\omega^n_{N,\chi,p^{m-\nu}}(S) \le k$) with $\nu := \nu_p(a^*(S))$. Moreover $\chi =' \chi_S$ holds.*

Proof. Note that all $a^*(S)$ lie in $\mathbb{Z}_{(p)}$, since they are obtained from the Fourier coefficients $a(T)$ by linear combinations with coefficients in $\mathbb{Z}$.

Seeking a contradiction, we suppose that there exists S such that the claim is not true. Let S_0 be one of S such that $\det S$ is minimal among such S. Then we consider

$$\begin{aligned}\phi_{S_0} - &\sum_{\substack{S \in \Lambda_r^+/\mathrm{GL}_r(\mathbb{Z}) \\ \det S < \det S_0}} A(S,S_0)\frac{a^*(S)}{\epsilon(S)}\theta_S^{(n)} \\ &\equiv \sum_{\substack{S \in \Lambda_r^+/\mathrm{GL}_r(\mathbb{Z}) \\ \det S \ge \det S_0}} A(S,S_0)\frac{a^*(S)}{\epsilon(S)}\theta_S^{(n)} \\ &\equiv a^*(S_0)\theta_{S_0}^{(n)} \bmod p^m.\end{aligned}$$

Here the last congruence follows from the fact that $S \not\sim S_0 \bmod \mathrm{GL}_r(\mathbb{Z})$ implies $A(S,S_0)=0$. Since $p \nmid \epsilon(S)$ by (2.1), we have $\frac{a^*(S)}{\epsilon(S)}\theta_S^{(n)}$ mod $p^m \in \widetilde{M}_k(\Gamma_0^{(n)}(N), \chi)_{p^m}$ for any $S \in \Lambda_r^+$ with $\det S < \det S_0$. It follows that

$$\phi_{S_0} - \sum_{\substack{S \in \mathrm{GL}_r(\mathbb{Z})\backslash\Lambda_r^+ \\ \det S < \det S_0}} A(S,S_0)\frac{a^*(S)}{\epsilon(S)}\theta_S^{(n)} \bmod p^m \in \widetilde{M}_k(\Gamma_0^{(n)}(N), \chi)_{p^m}.$$

This implies $a^*(S_0)\theta_{S_0}^{(n)}$ mod $p^m \in \widetilde{M}_k(\Gamma_0^{(n)}(N), \chi)_{p^m}$. This is a contradiction. Hence we have $a^*(S)\theta_S^{(n)}$ mod $p^m \in \widetilde{M}_k(\Gamma_0^{(n)}(N), \chi)_{p^m}$ for any $S \in \Lambda_r^+$. In particular if $a^*(S) \not\equiv 0$ mod p^m, then we have $\theta_S^{(n)}$ mod $p^{m-\nu} \in \widetilde{M}_k(\Gamma_0^{(n)}(N), \chi)_{p^{m-\nu}}$ with $\nu := \nu_p(a^*(S))$, and therefore $\omega^n_{N,\chi,p^{m-\nu}}(S) \le k$. The statement $\chi =' \chi_S$ follows from Proposition 2.4. □

For later use, we mention a simple consequence of Proposition 4.2 for the mod p case.

Corollary 4.3. *Let n, k, N be positive integers, r an even integer with $n \geq r$. Let p be an odd prime with $p > r+1$ and χ a quadratic Dirichlet character mod N with $\chi(-1) = (-1)^k$. Suppose that $F \in M_k(\Gamma_0^{(n+r)}(N), \chi)_{\mathbb{Z}_{(p)}}$ is mod p singular of p-rank r.*

(1) For any $S \in \Lambda_r^+$ with $a^*(S) \not\equiv 0$ mod p, we have $\widetilde{\theta_S^{(r)}} \in \widetilde{M}_k(\Gamma_0^{(r)}(N), \chi)$.
(2) Take $S \in \Lambda_r^+$ such that $\det S$ is minimal among those for which $a(S) \not\equiv 0$ mod p. Then we have $\widetilde{\theta_S^{(n)}} \in \widetilde{M}_k(\Gamma_0^{(n)}(N), \chi)$ and therefore $\omega_{N,\chi}^n(S) \leq k$.

Remark 4.4. The statement (2) can be proved also by the mod p version of Freitag's original arguments in [12]. Our strategy can be viewed as a refinement of his method.

Proof of Corollary 4.3. (1) By Proposition 4.2, we have $\widetilde{\theta_S^{(n)}} \in \widetilde{M}_k(\Gamma_0^{(n)}(N), \chi)$. Hence the claim follows from $\Phi^{n-r}(\theta_S^{(n)}) = \theta_S^{(r)}$ immediately.
(2) If $S \in \Lambda_r^+$ satisfies such the minimality condition, we have $a^*(S) = a(S)$ by the definition of $a^*(S)$. The claim follows from this fact. □

5 Description by theta series for the mod p case

In this section, we prove Theorem 3.2 (1) for $m = 1$. Therefore, we treat the mod p singular case in this section. The goal is to show that all strongly mod p singular modular forms are represented by a linear combination of finitely many theta series. In our method, two new tools are important: the existence of "abstract Sturm bounds" for detecting mod p singular forms and the refinement of Freitag's expansion as exposed in the previous section. The nice thing is that we do not need to consider the exact level of the theta series.

5.1 Abstract Sturm bounds

In general, "Sturm bounds" give an explicit finite set of $T \in \Lambda_n$ such that a modular form of degree n, with Fourier coefficients in $\mathbb{Z}_{(p)}$, is congruent to zero mod p if (and only if) all its Fourier coefficients at the elements of this set are congruent to zero mod p, see e.g., [22, 27]. Such an explicit finite set would usually contain quadratic forms of all ranks. We need a version, which involves only quadratic forms of rank $> r$ (finitely many) to detect mod p singular forms. We only need the existence of such a set, so we do not discuss explicit bounds for its size.

Let χ be a quadratic Dirichlet character mod N. Let $M_{k,r}^{p\text{-sing}}(\Gamma_0^{(n+r)}(N), \chi)$ be the submodule of $M_k(\Gamma_0^{(n+r)}(N), \chi)_{\mathbb{Z}_{(p)}}$ consisting of all mod p singular modular forms with p-rank $\leq r$. We denote by $\widetilde{M}_{k,r}^{p\text{-sing}}(\Gamma_0^{(n+r)}(N), \chi)$ the set of reduction mod p of elements of $M_{k,r}^{p\text{-sing}}(\Gamma_0^{(n+r)}(N), \chi)$. This is a subspace over $\mathbb{F}_p$ of the vector space $\widetilde{M}_k(\Gamma_0^{(n+r)}(N), \chi)$. We consider the quotient space

$$V = V_{k,r} := \widetilde{M}_k(\Gamma_0^{(n+r)}(N), \chi)/\widetilde{M}_{k,r}^{p\text{-sing}}(\Gamma_0^{(n+r)}(N), \chi).$$

Then we have $\dim V < \infty$, since $\dim \widetilde{M}_k(\Gamma_0^{(n+r)}(N), \chi) < \infty$. For fixed $T \in \Lambda_{n+r}$ with $\mathrm{rank}(T) > r$ we define a linear map $\ell_T : V \longrightarrow \mathbb{F}_p$ by

$$\ell_T(\widetilde{F} + \widetilde{M}^{p\text{-sing}}_{k,r}(\Gamma_0^{(n+r)}(N), \chi)) := \widetilde{a_F(T)}.$$

Clearly, the set L of all such ℓ_T is "total" for V, i.e., the intersection of the kernels of all ℓ_T is trivial. By linear algebra (V is of finite dimension!) we can choose a finite subset $\mathcal{T}_{n+r,r} = \{\ell_{T_1}, \ldots, \ell_{T_d}\}$ of L which is still total for V.

Conclusion: *In the situation above, there exist finitely many $T_1, \ldots, T_d$ with all T_j of rank larger than r such that for all $F \in M_k(\Gamma_0^{(n+r)}(N), \chi)_{\mathbb{Z}_{(p)}}$ the vanishing of $\widetilde{a_F(T_j)}$ for all T_j implies that F is mod p singular of p-rank $\leq r$.*

We call such $\mathcal{T}_{n+r,r}$ a "Sturm set" for mod p singular forms.

5.2 Proof of Theorem 3.2 (1) for $m = 1$

For the Sturm set $\mathcal{T}_{r,r-1}$ ($\subset \Lambda_r^+$) corresponding to $M^{p\text{-sing}}_{k,r-1}(\Gamma_0^{(r)}(N))$, we take a positive integer M such that

$$M > \max\{\det T \mid T \in \mathcal{T}_{r,r-1}\}.$$

We put

$$G := F - \sum_{\substack{S \in \Lambda_r^+/\mathrm{GL}_r(\mathbb{Z}) \\ \det S < M}} \frac{a^*(S)}{\epsilon(S)} \theta_S^{(n+r)}$$

and consider $g := \Phi^n(G)$. Note here that the summation over S is finite. Then we have

$$g = \Phi^n(F) - \sum_{\substack{S \in \Lambda_r^+/\mathrm{GL}_r(\mathbb{Z}) \\ \det S < M}} \frac{a^*(S)}{\epsilon(S)} \theta_S^{(r)}$$

and $\widetilde{g} \in \widetilde{M}_k(\Gamma_0^{(r)}(N), \chi)$ by Corollary 4.3 (1).

We prove now that $\widetilde{g} \in \widetilde{M}^{p\text{-sing}}_{k,r-1}(\Gamma_0^{(r)}(N), \chi)$. It suffices to check the Fourier coefficients $a_g(T)$ for all $T \in \mathcal{T}_{r,r-1}$. It follows from (4.3) that

$$a_g(T) = \sum_{\substack{S \in \Lambda_r^+/\mathrm{GL}_r(\mathbb{Z}) \\ \det S \geq M}} \frac{a^*(S)}{\epsilon(S)} A(S,T)$$

for any $T \in \Lambda_r^+$. If $S[X] = T$ with $X \in \mathbb{Z}^{r,r}$ then $\det S[X] = \det S (\det X)^2 = \det T$ and hence $\det S \leq \det T$. This implies that $A(S,T) = 0$ for T with $\det T < \det S$. Therefore we have $a_g(T) = 0$ for any $T \in \mathcal{T}_{r,r-1}$. This implies $\widetilde{g} \in \widetilde{M}^{p\text{-sing}}_{k,r-1}(\Gamma_0^{(r)}(N), \chi)$.

Suppose that $g \not\equiv 0 \bmod p$. Then g is mod p singular of some p-rank $r' < r$. By Theorem 2.3, we have $2k - r \equiv 2k - r' \equiv 0 \bmod p-1$ and hence $r' \equiv r \bmod$

$p-1$. Since $r' < r < p-1$, this is impossible. This shows $g \equiv 0 \bmod p$. Taking into account that F and G are mod p singular, we obtain that $G \equiv 0 \bmod p$. This completes the proof of (1) in Theorem 3.2. $\square$

Remark 5.1. The condition "$n \geq r$" was necessary to assure $\widetilde{g} \in \widetilde{M_k}(\Gamma_0^{(r)}(N), \chi)$.

6 Specification of levels

6.1 Proof of Theorem 3.2 (2) for $m = 1$

In the situation of Theorem 3.2 we have to specify the levels of $S \in \Lambda_r^+$ involved: To do so, we use Corollary 9.3, which follows from the modified q-expansion principle and Kitaoka's formula. We observe that $\widetilde{\theta_S^{(r)}} \in \widetilde{M_k}(\Gamma_0^{(r)}(N), \chi)$. Then Corollary 9.3 shows that the level of S is of the form requested and also the statement about the nebentypus follows. $\square$

Now we prove Theorem 3.6 in the remainder of this section. Namely, we discuss the case where the p-rank is 2 and the weight is the minimal one allowing mod p singular modular forms, which are not truly singular, i.e., $1 + \frac{p-1}{2}$.

6.2 Proof of Theorem 3.6

In Theorem 3.2, we assumed $n \geq r$. In this subsection, we discuss the special case of $n = 1$ and $r = 2$; this is the simplest case where the condition "strongly mod p singular" is violated. We use instead very special properties of binary quadratic forms.

Proposition 6.1. *Let p be an odd prime. Let $S \in \Lambda_2^+$ be of level $N = p^j N'$ with $p \nmid N'$. Suppose that $\theta_S^{(1)} \equiv \phi \bmod p$ for some $\phi \in M_k(\Gamma_1)_{\mathbb{Z}_{(p)}}$. Then $N' = 1$, i.e.,* $\mathrm{level}(S) = p^j$.

Remark 6.2. To include the case of $n = 1$, we should look at the degree 1 theta series $\theta_S^{(1)}$ and $\phi \in M_k(\Gamma_1)$.

Proof. Seeking a contradiction we suppose $N' > 1$. We apply Theorem 9.1 of Kitaoka's original result for $n = 1$ to $\theta_S^{(1)}$ with $M = \left(\begin{smallmatrix} * & * \\ p^j & N' \end{smallmatrix}\right)$. Then we have

$$\theta_S^{(1)}|M = \kappa \cdot \theta_{S'}^{(1)},$$

where S' is the same as in the proof of Proposition 9.2. Then $N'S'$ is half-integral, $(N', \mathrm{cont}(N'S')) = 1$, and $\mathrm{cont}(N'S') = p^\alpha$ when $p^\alpha \mid\mid \mathrm{cont}(S)$. Then $\frac{N'}{p^\alpha} S' \in \Lambda_2^+$ is primitive. Therefore we can take a prime l with $l \nmid N'$ such that

$$A\left(S', \frac{l \cdot p^\alpha}{N'}\right) = A\left(\frac{N'S'}{p^\alpha}, l\right) \not\equiv 0 \bmod p.$$

However, since ϕ is of level 1, we have also

$$A\left(S', \frac{l\cdot p^{\alpha}}{N'}\right) \equiv a_{\phi}\left(\frac{l\cdot p^{\alpha}}{N'}\right) = 0 \bmod p.$$

This contradicts and we get $N' = 1$. □

Hence level(S) is a power of p in the situation of Proposition 6.1. Then $\det(2S)$ also should be a power of p. By the elementary divisor theorem, we can find U, $V \in \mathrm{GL}_2(\mathbb{Z})$ such that

$$U(2S)V = \begin{pmatrix} p^i & 0 \\ 0 & p^{i+j} \end{pmatrix}.$$

Then we have $U(2S) = \left(\begin{smallmatrix} p^i & 0 \\ 0 & p^{i+j} \end{smallmatrix}\right) V^{-1}$ and hence

$$U(2S)^tU = p^i \begin{pmatrix} 1 & 0 \\ 0 & p^j \end{pmatrix} V^{-1t}U.$$

If we put $W := V^{-1t}U = \left(\begin{smallmatrix} w_1 & w_2 \\ w_3 & w_4 \end{smallmatrix}\right)$, then we can write

$$U(2S)^tU = p^i \begin{pmatrix} w_1 & w_3p^j \\ w_2 & w_4p^j \end{pmatrix}.$$

Since $U(2S)^tU$ is symmetric, we have $w_2 = w_3p^j$ and

$$U(2S)^tU = p^i \begin{pmatrix} w_1 & w_3p^j \\ w_3p^j & w_4p^j \end{pmatrix}.$$

Here, for later use, we assume that $\det(2S)$ is an odd power of p. Then j also should be odd and therefore we write j as $2j+1$. Putting $a := w_1/2$, $b := w_3p^j/2$, $d := w_4/2$, we have

$$S \sim S(i,j) = p^i \begin{pmatrix} a & bp^{j+1} \\ bp^{j+1} & dp^{2j+1} \end{pmatrix} \bmod \mathrm{GL}_2(\mathbb{Z})$$

with $adp - b^2p^2 = p$. From these argument, we may assume that $S = S(i,j)$ when we consider $\theta_S^{(1)}$ with $\det(2S)$ an odd power of p.

Now we try to show that high powers of p in the level of S imply high filtration weight of $\theta_S^{(1)}$:

Proposition 6.3. *Let p be a prime with $p \geq 5$ and $S \in \Lambda_2^+$. Suppose that* $\det(2S)$ *is an odd power of p. Then we have*

$$\omega(\theta_S^{(1)}) \geq p^{i+j} \cdot \frac{p+1}{2},$$

where i, j are determined by

$$S = S(i,j) = p^i \begin{pmatrix} a & bp^{j+1} \\ bp^{j+1} & dp^{2j+1} \end{pmatrix}$$

with $adp - b^2p^2 = p$.

Before proving this proposition, we confirm some notation and facts on filtration of modular forms mod p for degree 1 given by Serre [24] and Swwinerton-Dyer [28]. Let $f = \sum_{n=0}^{\infty} a_f(n)\boldsymbol{e}(nz)$ be a formal power series with $a_f(n) \in \mathbb{Z}_{(p)}$. We write $\omega(f)$ for $\omega_{1,1}^1(f)$. Let $V(p)$, $U(p)$ be the operators defined by

$$f|V(p) = \sum_{n=0}^{\infty} a_f(n)\boldsymbol{e}(pnz),$$
$$f|U(p) = \sum_{n=0}^{\infty} a_f(pn)\boldsymbol{e}(nz).$$

Then we have $\omega(f|V(p)) = p\omega(f)$, $\omega(f|U(p)) \leq \omega(f)$.

We put $\omega(S(i,j)) := \omega(\theta^{(1)}_{S(i,j)})$.

Proof of Proposition 6.3. Using the facts that $\theta^{(1)}_{S(i,j)} = \theta^{(1)}_{S(i-1,j)}|V(p)$ and $\omega(f|V(p)) = p\omega(f)$, we have

$$\omega(S(i,j)) = p^i \omega(S(0,j)).$$

Furthermore, by comparing the representation numbers of $A(S(0,j),pn)$ and $A(S(1,j-1),n)$, we can easily confirm that

$$\theta^{(1)}_{S(0,j)}|U(p) = \theta^{(1)}_{S(1,j-1)}.$$

Since $\omega(f) \geq \omega(f|U(p))$ in general, we have

$$\omega(S(0,j)) \geq \omega(S(1,j-1)) = p\omega(S(0,j-1)).$$

This implies

$$\omega(S(i,j)) \geq p^{i+j}\omega(S(0,0)) = p^{i+j}\omega(S(0,0)) = p^{i+j} \cdot \frac{p+1}{2}.$$

Here the last equality follows from $\omega(S(0,0)) = \frac{p+1}{2}$ and this is due to Theorem 2.7. This completes the proof. □

Corollary 6.4. *Let p be a prime with $p \geq 5$ and $S \in \Lambda_2^+$. Suppose that $\theta_S^{(1)} \equiv \phi$ mod p for some $\phi \in M_{\frac{p+1}{2}}(\Gamma_1)_{\mathbb{Z}_{(p)}}$. Then we have* $\mathrm{level}(S) = p$.

Proof. The assumption $\theta_S^{(1)} \equiv \phi$ mod p implies

$$\omega(S) \leq \frac{p+1}{2}. \tag{6.1}$$

On the other hand, the level of S is a power of p because of Proposition 6.1. Then $\det(2S)$ is a power of p. Note here that, actually in this case, the power of p is odd, because the quadratic character χ_S should have nontrivial p-component, otherwise $\frac{p+1}{2} \equiv 1 \bmod p-1$ should hold and this is impossible. Hence we can assume that $S = S(i,j)$. Then we have

$$\omega(S) = \omega(S(i,j)) \geq p^{i+j} \cdot \frac{p+1}{2}.$$

Combing this with (6.1), we have $i = j = 0$. Namely, we obtain $\mathrm{level}(S) = p$. □

We can now prove our result for the case of p-rank 2.

Proof of Theorem 3.6. Let $\{T_1, \cdots, T_{h_p}\} \subset \Lambda_2^+/\mathrm{GL}_2(\mathbb{Z})$ be a set of the representatives of $\mathrm{GL}_2(\mathbb{Z})$-equivalence classes of binary quadratic forms with level p. Note that $\theta_{T_i}^{(n+2)} \in M_1(\Gamma_0^{(n+2)}(p), (\frac{*}{p}))$, and there exists $G_i \in M_{\frac{p+1}{2}}(\Gamma_{n+2})$ such that $G_i \equiv \theta_{T_i}^{(n+2)} \bmod p$ by Theorem 2.7. Then G_i is mod p singular, because $\theta_{T_i}^{(n+2)}$ is true singular. Then we consider

$$H := F - \sum_{i=1}^{h_p} \frac{1}{2} a_F \begin{pmatrix} 0 & 0 \\ 0 & T_i \end{pmatrix} G_i \in M_{\frac{p+1}{2}}(\Gamma_{n+2})_{\mathbb{Z}_{(p)}}.$$

This H is a mod p singular of some p-rank $r' \leq 2$.

Now we suppose that still $r' = 2$. Then we can take $S \in \Lambda_2^+$ with $a_H\left(\begin{smallmatrix} 0 & 0 \\ 0 & S \end{smallmatrix}\right) \not\equiv 0 \bmod p$ such that $\det S$ is minimal. By Corollary 4.3 (2), we have $\widetilde{\theta_S^{(1)}} \in \widetilde{M}_{\frac{p+1}{2}}(\Gamma_1)$. By Corollary 6.4, we have $\mathrm{level}(S) = p$ and hence $S \sim T_j \bmod \mathrm{GL}_2(\mathbb{Z})$ for some j. However, from $a_H\left(\begin{smallmatrix} 0 & 0 \\ 0 & T_i \end{smallmatrix}\right) \equiv 0 \bmod p$ for any i, we have $a_H\left(\begin{smallmatrix} 0 & 0 \\ 0 & S \end{smallmatrix}\right) \equiv 0 \bmod p$. This is a contradiction.

Therefore we have $r' \leq 1$ or $H \equiv 0 \bmod p$. If $H \not\equiv 0 \bmod p$, then we have $p + 1 - r' \equiv 0 \bmod p-1$ because of Theorem 2.3. This is impossible, we therefore get $H \equiv 0 \bmod p$. This completes the proof of Theorem 3.6. □

Proof of Supplement 3.7. To prove $\Phi^{n+1}(F) \not\equiv 0 \bmod p$ it is sufficient to assure the mod p linear independence of the binary theta series in question: Let $\{T_1, \cdots, T_{h_p}\}$ be as in the proof of Theorem 3.6. As stated in Kani [15], $\theta_{T_j}^{(1)}$ $(j = 1, \cdots, h_p)$ are linearly independent over $\mathbb{C}$. Now we prove that $\theta_{T_j}^{(1)}$ $(j = 1, \cdots, h_p)$ are linearly independent over $\mathbb{F}_p$.

Suppose that $\sum_{i=1}^{h_p} c_i \theta_{T_j}^{(1)} \equiv 0 \bmod p$. Then, for any $n \geq 0$, we have

$$\sum_{i=1}^{h_p} c_i A(T_i, n) \equiv 0 \bmod p.$$

By Theorem 2.6 (1), we can find infinitely many primes l such that $A(T_i, l) > 0$ for each i. Then we have $A(T_j, l) = 0$ for any j with $j \neq i$, because of Theorem 2.6 (2). Since $A(T_i, l) = 2, 4, 6 \not\equiv 0 \bmod p$, we have $c_i \equiv 0 \bmod p$. This shows $c_i \equiv 0 \bmod p$ for any i with $1 \leq i \leq h_p$. Therefore $\theta_{T_j}^{(1)}$ $(j = 1, \cdots, h_p)$ are linearly independent over $\mathbb{F}_p$. This completes the proof of Supplement 3.7. □

7 From mod p to mod p^m

In this section, using induction, we extend Theorem 3.2 for $m = 1$ (proved in the previous sections) to the case of general m. The main reason why we prefer to give a mod p version first and then extend to arbitrary powers is the technical difficulty which one encounters, when one tries to give a mod p^m analogue of our abstract Sturm bounds. We avoid such a problem in this way.

We start with proving the following weaker version of Theorem 3.2.

Theorem 7.1 (Weaker version of Theorem 3.2)**.** *Let n, k, N be positive integers, r an even integer with $n \geq r$. Let p be an odd prime with $p > r+1$ and χ a quadratic Dirichlet character mod N with $\chi(-1) = (-1)^k$. Suppose that $F \in M_k(\Gamma_0^{(n+r)}(N), \chi)_{\mathbb{Z}_{(p)}}$ is mod p^m singular of p-rank r. Then there are finitely many $S \in \Lambda_r^+/\mathrm{GL}_r(\mathbb{Z})$ of level dividing $p^e N$ with $\widetilde{\theta_S^{(n)}} \in \widetilde{M}_k(\Gamma_0^{(n)}(N), \chi)$ (only mod p) such that*

$$F \equiv \sum_S c_S \theta_S^{(n+r)} \bmod p^m \quad (c_S \in \mathbb{Z}_{(p)}). \tag{7.1}$$

Moreover, all S involved satisfy $\chi =' \chi_S$.

Proof of Theorem 7.1. We prove the statement by induction on m. We have already proved the statement for $m = 1$. Suppose that the statements (1), (2) in Theorem 3.2 are true for any m with $m < m_0$. We consider the case $m = m_0$. Note that $2k - r \equiv 0 \bmod (p-1)p^{m_0-1}$ by Theorem 2.3. Therefore we can write $k = \frac{r}{2} + t \cdot \frac{p-1}{2} p^{m_0-1}$.

By the assumption, F is mod p^{m_0-1} singular (because of mod p^{m_0} singular) and therefore by the induction hypothesis, we have

$$F \equiv \sum_{\substack{S \in \Lambda_r^+ \\ \mathrm{level}(S) | p^{e_1} N}} c_S \theta_S^{(n+r)} \bmod p^{m_0-1} \quad \text{with} \quad c_S \in \mathbb{Z}_{(p)},$$

for some e_1, and all S involved satisfy $\chi =' \chi_S$.

We recall from [6] that for any prime p, there exists $E \in M_{\frac{p-1}{2}}(\Gamma_0^{(n+r)}(p), (\frac{*}{p}))_{\mathbb{Z}_{(p)}}$ satisfying $E \equiv 1 \bmod p$. We put $G := \sum_S c_S \theta_S^{(n+r)} \cdot E^{tp^{m_0-1}}$. Note here that F, $G \in M_k(\Gamma_0^{(n+r)}(p^{e_1} N), \chi)_{\mathbb{Z}_{(p)}}$ because of $\chi =' \chi_S$. Consider $H := \frac{1}{p^{m_0-1}}(F - G) \in M_k(\Gamma_0^{(n+r)}(p^{e_1} N), \chi)_{\mathbb{Z}_{(p)}}$. Then for any T, we have

$$a_F(T) = a_G(T) + p^{m_0-1} a_H(T).$$

Since $a_F(T) \equiv a_G(T) \equiv 0 \bmod p^{m_0}$ for any T with $\mathrm{rank}(T) > r$, we have $a_H(T) \equiv 0 \bmod p$ for any T with $\mathrm{rank}(T) > r$. This means that H is mod p singular of some p-rank $r' \leq r$. By $2k - r' \equiv 2k - r \equiv 0 \bmod p-1$ and $0 \leq r' \leq r \leq p-1$, we have

$r' = r$. Therefore $a_H(T) \not\equiv 0 \bmod p$ for some T with $\mathrm{rank}(T) = r$. Again by our theorem for $m = 1$, we have

$$H \equiv \sum_{\substack{R \in \Lambda_r^+ \\ \mathrm{level}(R) | p^{e_2} N}} d_R \theta_R^{(n+r)} \bmod p \quad \text{with} \quad d_R \in \mathbb{Z}_{(p)},$$

and all R involved satisfy $\chi =' \chi_R$. Noting $G := \sum_S c_S \theta_S^{(n+r)} \cdot E^{tp^{m_0-1}} \equiv \sum_S c_S \theta_S^{(n+r)} \bmod p^{m_0}$, we have

$$\begin{aligned} F &\equiv G + p^{m_0-1} H \\ &\equiv \sum_{\substack{S \in \Lambda_r^+ \\ \mathrm{level}(S) | p^{e_1} N}} c_S \theta_S^{(n+r)} + p^{m_0-1} \sum_{\substack{R \in \Lambda_r^+ \\ \mathrm{level}(R) | p^{e_2} N}} d_R \theta_R^{(n+r)} \bmod p^{m_0}. \end{aligned}$$

This completes the proof of Theorem 7.1. □

Completion of the proof of Theorem 3.2. We first claim that the coefficients c_S in (7.1) coincide mod p^m with $\frac{a^*(S)}{\epsilon(S)}$ provided that the summation in (7.1) is restricted to pairwise $\mathrm{GL}_r(\mathbb{Z})$-inequivalent elements of Λ_r^+. To see this, we take an arbitrary $S_0 \in \Lambda_r^+$. Then (using freely the notation from Section 4)

$$a(S_0) \equiv \sum_S c_S A(S, S_0) \bmod p^m$$

holds and passing to primitive Fourier coefficients gives

$$a^*(S_0) \equiv \sum_S c_S A^*(S, S_0) \bmod p^m.$$

Now $A^*(S, S_0) = \epsilon(S_0)$ if S and S_0 are $\mathrm{GL}_r(\mathbb{Z})$-equivalent and zero otherwise; this proves the claim from above.

To complete the proof of Theorem 3.2, we have to prove $\widetilde{\theta_S^{(n)}} \in \widetilde{M}_k(\Gamma_0^{(n)}(N), \chi)_{p^{m-\nu}}$; this follows now directly from Proposition 4.2. This completes the proof of Theorem 3.2. □

8 Appendix A: Modified q-expansion principle

To specify the level of theta series in our theorems, we need some control over the behaviour of congruences of modular forms, when we switch to other cusps. The "q-expansion principles" available in the literature do not exactly provide the information necessary for our purpose.

We fix a prime p and a positive integer N with $p \nmid N$. For $m \geq 0$ let R_m be a subring of $\mathbb{C}$ containing $\mathbb{Z}[e^{\frac{2\pi i}{N \cdot p^m}}, e^{\frac{2\pi i}{3}}, \frac{1}{6}, \frac{1}{N}]$.

Theorem 8.1. *Let p be an odd prime and N a positive integer with $p \nmid N$.*

(1) Let $f \in M_k(\Gamma_0^{(n)}(p^m) \cap \Gamma^{(n)}(N))$. Then for all $\gamma \in \Gamma_0^{(n)}(p^m)$ we have

$$f \in M_k(\Gamma_0^{(n)}(p^m) \cap \Gamma^{(n)}(N))_{R_m} \iff f|_k\gamma \in M_k(\Gamma_0^{(n)}(p^m) \cap \Gamma^{(n)}(N))_{R_m}.$$

(2) The same statement as in (1) holds for $f \in M_k(\Gamma_0^{(n)}(p^m) \cap \Gamma^{(n)}(N), (\frac{*}{p}))$.

The case $m = 0$ is the statement of the usual q-expansion principle in the formulation of Pitale et al. [21] proved by Katz [17] for $n = 1$ and Ichikawa [14] for $n > 1$. The main point in our version is that f and $f|\gamma$ share the same p-integrality property. Note that one only needs to prove one direction of the theorem. The aim of Appendix A is to prove this property.

For this aim, we need the existence of modular forms $\mathcal{E}_0 \in M_{l_0}(\Gamma_0^{(n)}(p))_{\mathbb{Z}_{(p)}}$ and $\mathcal{E}_1 \in M_{l_1}(\Gamma_0^{(n)}(p), (\frac{*}{p}))_{\mathbb{Z}_{(p)}}$ such that

$$\mathcal{E}_i \equiv 1 \bmod p,$$
$$\mathcal{E}_i|\omega_j \equiv 0 \bmod p \quad (1 \le j \le n).$$

Here ω_j can be any element $\left(\begin{smallmatrix} * & * \\ C & D \end{smallmatrix}\right) \in \Gamma_n$ with C being of rank i in $M_n(\mathbb{F}_p)$. We mention that the congruence condition of $\mathcal{E}_i$ can be rephrased as saying that for any $\gamma \in \Gamma_n$, we have

$$\mathcal{E}_i|\gamma \equiv 1 \bmod p \quad \iff \quad \gamma \in \Gamma_0^{(n)}(p)$$

and $\mathcal{E}_i|\gamma \equiv 0 \bmod p$ for $\gamma \notin \Gamma_0^{(n)}(p)$.

The existence of such $\mathcal{E}_0$ was explained in [3], under the condition $p \ge n + 3$. However the reasoning there should be simplified. The only assumption is the existence of a modular form E of degree n with level 1 such that $E \equiv 1 \bmod p$. This is assured by the condition $p > n + 3$ in Böcherer-Nagaoka [6]. But the condition can be made weaker, if one does not demand such E to be of weight $p - 1$.

For a prime p with $p \equiv 1 \bmod 4$, one can use the existence of a p-special even unimodular quadratic form (rank $2p-2$) and its theta series; here "p-special" means that S has an automorphism of order p without nontrivial fixpoints. For a prime p with $p \equiv -1 \bmod 4$, such quadratic form exists for rank $4p - 4$. Both existence results are shown in Dummigan-Tiep [10]. So such E exists for any prime p (even for $p = 2$). Using this fact, now we can prove the following statement which is an improvement of Theorem III c) in [3].

Theorem 8.2. *For any odd prime p, there exist $\mathcal{E}_i$ as above for some l_i ($i = 0$, 1).*

Proof. We prove the existence of $\mathcal{E}_0$. We put

$$H_j := E - p^{l_0 j} E(p^j z) \quad (z \in \mathbb{H}_n).$$

It follows that

$$\begin{aligned} H_h|\omega_j &\equiv E \equiv 1 \bmod p \quad \text{for} \quad h < j, \\ H_j|\omega_j &\equiv 0 \bmod p. \end{aligned}$$

Then one takes products of these H_j to get the function $\mathcal{E}_0$.

As for the the existence of $\mathcal{E}_1$, we take just $\mathcal{E}_0$ from above and a theta series $\theta_S^{(n)}$ of level p with quadratic nebentypus $(\frac{*}{p})$ attached to the quadratic form S corresponding to the p-special lattice. Then for a suitable $t \geq 1$,

$$\mathcal{E}_1 := \mathcal{E}_0^t \cdot \theta_S^{(n)}$$

is nebentypus analogue for $\mathcal{E}_0$; t has to be large enough to kill denominators of $\theta_S^{(n)}|\omega_j$. □

8.1 Proof of Theorem 8.1 (1) for $m = 1$

We prove Theorem 8.1 (1) for $m = 1$. We take $\mathcal{E} := \mathcal{E}_0$ from Theorem 8.2. Taking an appropriate power α of $\mathcal{E}$, we consider a trace from level $\Gamma_0^{(n)}(p) \cap \Gamma^{(n)}(N)$ to level $\Gamma^{(n)}(N)$:

$$F = f\mathcal{E}^\alpha + \sum_{j \bmod p} (f\mathcal{E}^\alpha)|\delta_j$$

where δ_j are representatives of the left cosets not in $\Gamma_0^{(n)}(p) \cap \Gamma^{(n)}(N)$, in particular, $\mathcal{E}|\delta_j \equiv 0 \bmod p$. Applying γ, we see that $\delta_j \cdot \gamma \notin \Gamma_0^{(n)}(p)$ and hence $(f\mathcal{E}^\alpha)|\delta_j \cdot \gamma \equiv 0 \bmod p$. We observe that F is of level $\Gamma^{(n)}(N)$ and hence we may apply the ordinary q-expansion principle: We have

$$F|\gamma \equiv (f|\gamma)\mathcal{E}^\alpha \equiv f|\gamma \bmod p.$$

In particular $F|\gamma$ is p-integral and hence $f|\gamma$ is also p-integral. This completes the proof of (1) for $m = 1$. □

8.2 Proof of Theorem 8.1 (1) for $m \geq 2$

We prove Theorem 8.1 (1) for $m \geq 2$ by induction on m. We suppose that the statement is true for the level $\Gamma_0^{(n)}(p^{m-1})$. We put

$$\mathcal{F} := \mathcal{E}(p^{m-1}z) = p^{-(m-1)\frac{nl}{2}}\mathcal{E}| \begin{pmatrix} p^{m-1} \cdot 1_n & 0_n \\ 0_n & 1_n \end{pmatrix} \in M_l(\Gamma_0^{(n)}(p^m))_{\mathbb{Z}_{(p)}}.$$

Here l is the weight of $\mathcal{E} = \mathcal{E}_0$ and $z \in \mathbb{H}_n$ is the variable of $\mathcal{E}$. Taking some positive integer α, we consider $f\mathcal{F}^\alpha$ and the trace of this from $\Gamma_0^{(n)}(p^m) \cap \Gamma^{(n)}(N)$ to

$\Gamma_0^{(n)}(p^{m-1}) \cap \Gamma^{(n)}(N)$;

$$F = \mathrm{tr}(f\mathcal{F}^\alpha) = f\mathcal{F}^\alpha + \sum_{w}^{p-1}(f\mathcal{F}^\alpha)|\gamma_w \tag{8.1}$$
$$= f\mathcal{F}^\alpha + \sum_{j=1}^{p-1}(f|\gamma_w)(\mathcal{F}|\gamma_w)^\alpha,$$

where $\gamma_w = \left(\begin{smallmatrix} 1_n & 0_n \\ p^{m-1}N_w & 1_n \end{smallmatrix}\right)$ and w runs over a complete set of representatives of integral symmetric matrices of size n mod p except the trivial coset $p \cdot \mathrm{Sym}_n(\mathbb{Z})$. Now we prove that $(f|\gamma_w)(\mathcal{F}|\gamma_w)^\alpha \equiv 0$ mod p. By a direct calculation, we have

$$\begin{pmatrix} p^{m-1}\cdot 1_n & 0_n \\ 0_n & 1_n \end{pmatrix}\begin{pmatrix} 1_n & 0_n \\ p^{m-1}Nw & 1_n \end{pmatrix} = \begin{pmatrix} 1_n & 0_n \\ Nw & 1_n \end{pmatrix}\begin{pmatrix} p^{m-1}\cdot 1_n & 0_n \\ 0_n & 1_n \end{pmatrix}. \tag{8.2}$$

This implies

$$\mathcal{F}|\begin{pmatrix} 1_n & 0_n \\ p^{m-1}Nw & 1_n \end{pmatrix} = p^{-(m-1)\frac{nl}{2}}\mathcal{E}|\begin{pmatrix} p^{m-1}\cdot 1_n & 0_n \\ 0_n & 1_n \end{pmatrix}\begin{pmatrix} 1_n & 0_n \\ p^{m-1}Nw & 1_n \end{pmatrix}$$
$$= p^{-(m-1)\frac{nl}{2}}\mathcal{E}|\begin{pmatrix} 1_n & 0_n \\ Nw & 1_n \end{pmatrix}\begin{pmatrix} p^{m-1}\cdot 1_n & 0_n \\ 0_n & 1_n \end{pmatrix}.$$

Since $\mathcal{E}|\left(\begin{smallmatrix} 1_n & 0_n \\ Nw & 1_n \end{smallmatrix}\right) \equiv 0$ mod p and we can put $\mathcal{E}|\left(\begin{smallmatrix} 1_n & 0_n \\ Nw & 1_n \end{smallmatrix}\right) = pX$ for some p-integral modular form $X = X_w$. Then we have

$$\mathcal{F}|\begin{pmatrix} 1_n & 0_n \\ p^{m-1}Nw & 1_n \end{pmatrix} = p^{-(m-1)\frac{nl}{2}}\mathcal{E}|\begin{pmatrix} 1_n & 0_n \\ Nw & 1_n \end{pmatrix}\begin{pmatrix} p^{m-1}\cdot 1_n & 0_n \\ 0_n & 1_n \end{pmatrix}$$
$$= p^{-(m-1)\frac{nl}{2}}(pX)|\begin{pmatrix} p^{m-1}\cdot 1_n & 0_n \\ 0_n & 1_n \end{pmatrix}$$
$$= pX(p^{m-1}z) \equiv 0 \bmod p.$$

Therefore $(f|\gamma_w)(\mathcal{F}|\gamma_w)^\alpha \equiv 0$ mod p follows if α is large. By the assumption that f is p-integral, F is also p-integral.

Now let $\delta \in \Gamma_0^{(n)}(p^m)$, we may apply induction because F is of level $\Gamma_0^{(n)}(p^{m-1}N)$. We start form (8.1) for F and apply δ to it: If f is p-integral, then F is also p-integral, and then $F|\delta$ is also p-integral. By (8.1), we have

$$F|\delta = (f|\delta)\mathcal{F}^\alpha + \sum_{j=1}^{p-1}(f\mathcal{F}^\alpha)|\gamma_w\delta$$

The first term on the right hand side of (8.1) is congruent mod p to $f|\delta$. We prove $(f\mathcal{F}^\alpha)|\gamma_w\delta \equiv 0$ mod p. To prove this, we must multiply (8.2) from the right by $\delta = \left(\begin{smallmatrix} a & b \\ p^m c & d \end{smallmatrix}\right)$. We get

$$\begin{pmatrix} p^{m-1}\cdot 1_n & 0_n \\ 0_n & 1_n \end{pmatrix}\gamma_w\delta = \begin{pmatrix} a & p^{m-1}b \\ wNa + pc & wNp^{m-1}b + d \end{pmatrix}\begin{pmatrix} p^{m-1}\cdot 1_n & 0_n \\ 0_n & 1_n \end{pmatrix}.$$

Since $p \nmid wNa$ and by a similar argument as above, we obtain $\mathcal{F}|\gamma_w\delta \equiv 0 \bmod p$. This shows $(f\mathcal{F}^\alpha)|\gamma_w\delta \equiv 0 \bmod p$ for large α. This implies that $F|\delta \equiv f|\delta \bmod p$ and hence $f|\delta$ is p-integral. □

8.3 Proof Theorem 8.1 (2)

A minor variant of the proof above applies if $f \in M_k(\Gamma_0^{(n)}(p^m) \cap \Gamma^{(n)}(N), (\frac{*}{p}))$. The proof above needs to be modified only for the step "$m = 1$", where one has to use a function $\mathcal{E} = \mathcal{E}_1$ of Theorem 8.2. □

9 Appendix B: Kitaoka's formula

In this Appendix, we generalize Kitaoka's theorem to higher degree to specify the level of theta series in our theorem. We then apply this in practice and discuss levels.

9.1 Generalization of Kitaoka's formula

We freely switch here between the language of (positive integral) quadratic forms S (or integral symmetric matrices) and the language of lattices L in an euclidean space $(\mathbb{R}^m, \langle\ ,\ \rangle)$. The aim is to get a degree n version of a result of Kitaoka; we state it here only in the version sufficient for our application and allowing a rather short proof. A more sophisticated version generalizing the computation of Kitaoka would lead to a more general statement.

For $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$, we denote by $\mathfrak{M}$ the n-fold diagonal embedding of M into $\mathrm{Sp}_n(\mathbb{R})$, namely, $\mathfrak{M} := \left(\begin{smallmatrix} a\cdot 1_n & b\cdot 1_n \\ c\cdot 1_n & d\cdot 1_n \end{smallmatrix}\right)$.

Theorem 9.1. *Let d be a positive divisor of N with $(d, \frac{N}{d}) = 1$, and $c := \frac{N}{d}$. Choose integers a, b such that $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Then, there exists a constant κ such that for any an even integral positive definite lattice L of rank $m = 2k$ and level N, one has*

$$\theta_L^{(n)}|_k\, \mathfrak{M} = \kappa \cdot \theta_{L'}^{(n)}, \tag{9.1}$$

where L' is a lattice characterized by

$$L' \otimes \mathbb{Z}_p = \begin{cases} L \otimes \mathbb{Z}_p & \text{if } p \nmid d, \\ (L \otimes \mathbb{Z}_p)^* & \text{if } p \mid d, \end{cases}$$

and "$*$" denotes the dual.

Before we begin the proof, we need to prepare some more notation and recall the facts: Kitaoka [18] proved the degree one version of this statement for a more general type of theta series

$$\theta_L^{(1)}(q)(z) = \sum_{\mathfrak{x}\in L} q(\mathfrak{x}) e^{2\pi i \langle \mathfrak{x},\mathfrak{x}\rangle \cdot z}$$

allowing homogeneous harmonic polynomials q on $\mathbb{C}^m$ as coefficients in the theta series. An inspection of his proof shows that the constants of proportionality do NOT depend on those harmonic polynomials.

Ibukiyama [13] investigated holomorphic differential operators $\mathcal{D}$ on $\mathbb{H}_n$, which are polynomials in the partial holomorphic derivatives, evaluated in $\mathrm{diag}(z_1, \cdots, z_n)$; Such a $\mathcal{D}$ maps holomorphic functions F on $\mathbb{H}_n$ to holomorphic functions $\mathcal{D}(F)$ defined on $\mathbb{H}_1^n$ such that, with the notation above

$$\mathcal{D}(F|_k \mathfrak{M}) = \mathcal{D}(F)|_{k_1}^{(z_1)} M \cdots |_{k_n}^{(z_n)} M$$

holds for arbitrary $M \in \mathrm{SL}_2(\mathbb{R})$ with certain weights $k_i \geq k$. The upper index z_i at the slash operator indicates that $|_{k_i} M$ should be applied w.r.t. the variable z_i.

He also showed that from all the $\mathcal{D}(F)$ one can recover the Taylor expansion of F with respect to the non-diagonal coefficients of $Z = (z_{ij}) \in \mathbb{H}_n$. To show the theorem, it is then enough, to show the equality of both sides after applying all such differential operators.

To do this we need some more notation: For a fixed differential operator $\mathcal{D}$ of Ibukiyama type and $w_1, \cdots, w_n \in \mathbb{C}^m$ changing the automorphy factor from $\det^k$ to $k_1, \cdots, k_n$ in the variables $z_1, \cdots, z_n$ we define

$$\mathcal{D} e^{2\pi i \sum \langle w_i, w_j\rangle z_{ij}} = P(w_1, \cdots, w_n) e^{2\pi i \sum \langle w_i, w_i\rangle z_i}$$

with $z_i := z_{ii}$. Then P is a harmonic polynomial in each of the variables w_i and we may write it as a finite sum of pure tensors:

$$P(w_1, \cdots, w_n) = \sum_t q_{1,t}(w_1) \cdots q_{n,t}(w_n),$$

where the $q_{i,t}$ are harmonic polynomials in the variables w_i.

Proof of Theorem 9.1. Applying $\mathcal{D}$ to the right hand side of (9.1) gives

$$\kappa \cdot \sum_t \theta_{L'}^{(1)}(q_{1,t})(z_1) \cdots \theta_{L'}^{(1)}(q_{n,t})(z_n).$$

Applying $\mathcal{D}$ to the left hand side gives

$$\sum_t \theta_L^{(1)}(q_{1,t})(z_1) \cdots \theta_L^{(1)}(q_{n,t})(z_n)|_{k_1} M \cdots |_{k_n} M.$$

Now one has to use Kitaoka's result in degree 1 to get our assertion. □

9.2 Application

Kitaoka's formula allows us to exclude certain congruences between modular forms and theta series of higher level (as far as the prime to p components of the levels are concerned):

Proposition 9.2. *Let n be an even positive integer and p a prime with $p > n+1$. Let N, N' be coprime to p and $S \in \Lambda_n^+$ with level $p^a N$. Assume that*

$$p^b N' \mid p^a N \quad \text{and} \quad \phi \equiv \theta_S^{(n)} \bmod p$$

for some $\phi \in M_k(\Gamma_0^{(n)}(p^b N'), \chi)_{\mathbb{Z}_{(p)}}$ with $\chi^2 = 1$. Then $N = N'$ and $\chi_N = (\chi_S)_N$.

Proof. We want to apply the modified q-expansion principle. To do so, we must modify ϕ and $\theta_S^{(n)}$ to arrive at the same weights and same (quadratic) p-component of nebentypus: We take $E \in M_{\frac{p-1}{2}}(\Gamma_0^{(n)}(p), (\frac{*}{p}))$ from Böcherer-Nagaoka [6] such that $E \equiv 1 \bmod p$. Then Proposition 2.4 allows us to choose $t \in \mathbb{N}$ such that $\theta_S^{(n)} \cdot E^t$ and ϕ have the same weight and the same p-component of nebentypus (this holds if $k \geq \frac{n}{2}$; the modification for the case of the opposite inequality is obvious). Then we can indeed apply Theorem 8.1 to $\frac{1}{p}\left(\phi - \theta_S^{(n)} \cdot E^t\right)$.

We may enlarge b (if necessary) to get $a = b$. Assume that there is a prime q different from p with $q^r || N$ and $q^s || N'$ with $s < r$. We apply Kitaoka's theorem to $\theta_S^{(n)}$ with

$$M = \begin{pmatrix} * & * \\ p^a \cdot \frac{N}{q^r} & q^r \end{pmatrix}.$$

By the modified q-expansion principle Theorem 8.1, we have $\theta_S^{(n)}|\mathfrak{M} \equiv \pm\phi|\mathfrak{M} \bmod p$, where the $\pm$ comes in from applying M to E^t.

Let L correspond to S and correspond L' to S'. Then S' is a rational symmetric matrix with q^r appearing in its denominator. The Fourier expansion of $\theta_S^{(n)}|\mathfrak{M} = \kappa \cdot \theta_{S'}^{(n)}$ has a S'th Fourier coefficient which is

$$\kappa \cdot A(S', S').$$

Since $p > n+1$, we see from (2.1) that $A(S', S') \not\equiv 0 \bmod p$. On the other hand, $\phi|\mathfrak{M}$ does not have a nonzero Fourier coefficient at S', because the q-part of the width of the cusp $\mathfrak{M}$ for $\Gamma_0^{(n)}(p^a N')$ is q^s with $s < r$. Therefore, $\kappa \cdot A(S', S')$ and hence κ must be divisible by p, i.e., $\kappa \in p \cdot R_{p^a}$. But the modified q-expansion principle tells us that $\theta_{S'}^{(n)}|\mathfrak{M}^{-1} = \frac{1}{\kappa} \cdot \theta_S^{(n)}$ is p-integral, in particular $\frac{1}{\kappa} \in R_{p^a}$. This is a contradiction, and we obtain $N' = N$.

The statement about the N-components of nebentypus characters follows by applying the modified q-expansion principle Theorem 8.1 with any integral matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ satisfying $C \equiv 0_n \bmod p^r N$ and $\det D \equiv 1 \bmod p$. □

Corollary 9.3. *Let n be a positive even integer and p a prime with $p > n+1$. Let $S \in \Lambda_n^+$. Assume that $\theta_S^{(n)} \equiv \phi \bmod p$ for some $\phi \in M_k(\Gamma_0^{(n)}(N), \chi)_{\mathbb{Z}_{(p)}}$ with $\chi^2 = 1$. Then the level of S is of the form "p-power $\times N'$" with some $N' \mid N$ and $\chi =' \chi_S$.*

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