整数のFourier係数をもつ次数2のHermiteモジュラー形式環の構造について

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§1. Introduction

Introduction

- $\Gamma_1 := SL_2(\mathbb{Z})$,
- e_4, e_6: the normalized Eisenstein series of weight 4, 6 for Γ_1, i.e.,

$$e_4 = 1 + 240 \sum_{n=1}^{\infty} \left(\sum_{0<d|n} d^3 \right) q^n, \quad e_6 = 1 - 504 \sum_{n=1}^{\infty} \left(\sum_{0<d|n} d^5 \right) q^n,$$

- δ: the Ramanujan delta function, i.e.,

$$\begin{aligned} \delta &:= 2^{-6} \cdot 3^{-3} (e_4^3 - e_6^2) \\ &= q - 24q^2 + 252q^3 + \cdots . \end{aligned}$$

Classical Result

- $M_k(\Gamma_1;\mathbb{Z})$: $\mathbb{Z}$-module consisting of modular forms of weight k for Γ_1 s.t. Fourier coefficients are in $\mathbb{Z}$.

Define

$$A(\Gamma_1;\mathbb{Z}) := \bigoplus_{k\in\mathbb{Z}} M_k(\Gamma_1;\mathbb{Z}) \quad \text{(algebra / } \mathbb{Z}\text{)}.$$

Classical Result:

$$A(\Gamma_1;\mathbb{Z}) = \mathbb{Z}[e_4, e_6, \delta].$$

How about the case of several variables?

- Siegel modular forms $\cdots$ Igusa's result 1979
- Hermitian modular forms $\cdots$ unknown!

Today's Talk:
A result on Hermitian modular forms over $\mathbb{Q}(\sqrt{-1})$.

§2. Preliminaries

Hermitian modular forms (of degree 2)

- Base field: $\boldsymbol{K} := \mathbb{Q}(\sqrt{-1})$, $\mathcal{O} := \mathbb{Z}[\sqrt{-1}]$,

- Hermitian upper half space:

$$\mathbb{H}_2 := \{ Z \in M_2(\mathbb{C}) \mid \tfrac{1}{2i}(Z - {}^t\overline{Z}) > 0 \text{ (pos. def.)} \},$$

- Hermitian modular group:

$$U_2(\mathcal{O}) := \{ \, M \in M_4(\mathcal{O}) \mid {}^t\overline{M} J_2 M = J_2 \} \; \left(J_2 := \begin{pmatrix} 0_2 & -1_2 \\ 1_2 & 0_2 \end{pmatrix}\right)$$

$$= \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid A{}^t\overline{D} - B{}^t\overline{C} = 1_2, \; A{}^t\overline{B}, \; C{}^t\overline{D} \in \mathrm{Her}_2(\mathcal{O}) \right\},$$

Generalized fractional transformation

- Generalized fractional transformation:

$$M\langle Z\rangle := (AZ+B)(CZ+D)^{-1},$$
$$Z \in \mathbb{H}_2, \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U_2(\mathcal{O}).$$

Definition of Hermitian modular forms

$F : \mathbb{H}_2 \longrightarrow \mathbb{C}$: holomorphic function,

- F: **Hermitian modular form** of weight k for $U_2(\mathcal{O})$ (with character $\det^{k/2}$)

$$\overset{\text{def}}{\iff} \quad F(M\langle Z\rangle) = \det(M)^{\frac{k}{2}} \cdot \det(CZ+D)^k F(Z),$$
$$\text{for} \quad \forall M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U_2(\mathcal{O}).$$

- F: **Symmetric** (resp. **Skew symmetric**)

$$\overset{\text{def}}{\iff} \; F({}^tZ) = F(Z) \text{ (resp. } F({}^tZ) = -F(Z)),$$

Some remarks

- $M_k(U_2(\mathcal{O}))\ (= M_k^{\mathrm{Sym}}(U_2(\mathcal{O}), \det^{k/2}))$

$$:=\{ \text{ symmetric Hermitian modular forms}$$

$$\text{of weight } k \text{ with character } \det^{k/2} \ \},$$

Remarks:

(1) $\forall M \in U_2(\mathcal{O})$, $\exists \varepsilon \in \mathcal{O}^\times$ s.t. $\det M = \varepsilon^2$,

(2) $4 \mid k \implies \det^{k/2} = \mathbf{1}$,

(3) k: odd $\implies M_k(U_2(\mathcal{O})) = \{0\}$.

Fourier expansion

- $F \in M_k(U_2(\mathcal{O})) \Longrightarrow$

$$F(Z) = \sum_{0 \le H \in \Lambda_2(\boldsymbol{K})} a_F(H) q^H, \quad q^H := e^{2\pi i \mathrm{tr}(HZ)}, \quad Z \in \mathbb{H}_2.$$

Here H runs over all positive semi-definite elements of

$$\Lambda_2(\boldsymbol{K}) := \{ H = (h_{ij}) \in \mathrm{Her}_2(\boldsymbol{K}) \mid h_{ii} \in \mathbb{Z},\ 2h_{ij} \in \mathcal{O} \}.$$

- $R \subset \mathbb{C}$: ring,

$$M_k(U_2(\mathcal{O}); R) := \{ F = \sum_{H \in \Lambda_2} a_F(H) q^H \mid a_F(H) \in R \ (\forall H \in \Lambda_2(\boldsymbol{K})) \}.$$

Hermitian Eisenstein series

- $k \geq 4$: even

$$E_k(Z) := \sum_{M=\left(\begin{smallmatrix} * & * \\ C & D \end{smallmatrix}\right)} (\det M)^{\frac{k}{2}} \det(CZ+D)^{-k}, \quad Z \in \mathbb{H}_2$$

$$\Longrightarrow E_k \in M_k(U_2(\mathcal{O}); \mathbb{Q}).$$

Here $M = \begin{pmatrix} * & * \\ C & D \end{pmatrix}$ runs over the representatives of $\left\{ \begin{pmatrix} * & * \\ 0_2 & * \end{pmatrix} \right\} \backslash U_2(\mathcal{O})$.

- $a_{E_k}(H)$ can be calculated by Krieg's formula!

Krieg's formula

$$a_{E_k}(H) = \begin{cases} 1 & \text{if } H = 0_2, \\ -\dfrac{2k}{B_k}\,\sigma_{k-1}(\varepsilon(H)) & \text{if } \operatorname{rk}(H) = 1, \\ \dfrac{4k(k-1)}{B_k \cdot B_{k-1,\chi_{-4}}} \displaystyle\sum_{0<d\mid\varepsilon(H)} d^{k-1} G_{\boldsymbol{K}}(k-2, 4\det(H)/d^2) & \text{if } \operatorname{rk}(H) = 2, \end{cases}$$

$$\varepsilon(H) := \max\{l \in \mathbb{N} \mid l^{-1}H \in \Lambda_2(\boldsymbol{K})\},$$

$$G_{\boldsymbol{K}}(m, N) := \frac{1}{1 + |\chi_{-4}(N)|}(\sigma_{m,\chi_{-4}}(N) - \sigma^*_{m,\chi_{-4}}(N)),$$

$$\sigma_{m,\chi_{-4}}(N) := \sum_{0<d\mid N} \chi_{-4}(d) d^m, \quad \sigma^*_{m,\chi_{-4}}(N) := \sum_{0<d\mid N} \chi_{-4}(N/d) d^m.$$

Structure theorem over **Z**[1/2,1/3]

We put $H_4 := E_4$,

$$H_8 := -\frac{61}{2^{10}\cdot 3^2\cdot 5^2}(E_8 - H_4^2),$$

$$F_{10} := -\frac{277}{2^9\cdot 3^3\cdot 5^2\cdot 7}(E_{10} - H_4E_6),$$

$$H_{12} := -\frac{19\cdot 691\cdot 2659}{2^{11}\cdot 3^7\cdot 5^3\cdot 7^2\cdot 73},$$

$$\times\left(E_{12} - \frac{3^2\cdot 7^2}{691}H_4^3 - \frac{2\cdot 5^3}{691}E_6^2 + \frac{2^9\cdot 3^4\cdot 5^2\cdot 7^2\cdot 6791}{19\cdot 691\cdot 2659}H_4H_8\right).$$

Structure theorem over **Z**[1/2,1/3]

Define

$$A^{(m)}(U_2(\mathcal{O});R) := \bigoplus_{k\in m\mathbb{Z}} M_k(U_2(\mathcal{O});R) \quad \text{(algebra/}R\text{)}.$$

Theorem (Nagaoka-K):

(1) All Fourier coefficients of H_4, E_6, H_8, F_{10}, H_{12} are in $\mathbb{Z}$,

(2) we have

$$A^{(2)}(U_2(\mathcal{O});\mathbb{Z}[1/2,1/3]) = \mathbb{Z}[1/2,1/3][H_4, E_6, H_8, F_{10}, H_{12}].$$

Remark: 5 modular forms are alg. indep. over $\mathbb{C}$, by Dern-Krieg's result.

§3. Main Results

Construction of generators over $\mathbf{Z}$

We set

$$I_{12} := 2^{-6} \cdot 3^{-3}(H_4^3 - E_6^2) + 2^4 \cdot 3^2 H_{12},$$
$$J_{12} := E_6^2,$$
$$H_{16} := 2^{-1} \cdot 3^{-1}(E_6 F_{10} - H_4^2 H_8)$$
$$I_{16} := 2^{-2} \cdot 3^{-1}(H_4 H_{12} - H_{16}),$$
$$H_{20} := 2^{-2} \cdot 3^{-2}(F_{10}^2 - H_4 H_8^2 - 2^2 \cdot 3 H_8 H_{12}),$$
$$H_{24} := 2^{-3} \cdot 3^{-1}(H_{12}^2 - H_4 H_{20}) - 2^{-1} \cdot 3^{-1} H_8 I_{16}.$$

Construction of generators over **Z**

In order to construct further generators, we use K.

$$K_{14} := 2^{-1} \cdot 3^{-1}(H_4F_{10} - E_6H_8),$$
$$K_{18} := 2^{-2} \cdot 3^{-1}(E_6H_{12} - H_4K_{14}),$$
$$K_{22} := 2^{-1} \cdot 3^{-1}(F_{10}H_{12} - H_8K_{14}),$$
$$K_{30} := 2^{-1} \cdot 3^{-1}(E_6H_{24} - K_{14}I_{16}) + 3^{-1}H_8F_{10}I_{12},$$
$$K_{42} := 2^{-2} \cdot 3^{-1}(H_{12}K_{30} - K_{14}H_{28}) - 2^{-1}H_8I_{12}K_{22}.$$

Construction of generators over **Z**

Finally we put

$$
\begin{aligned}
I_{24} &:= E_6 K_{18}, \quad H_{28} := 2^{-1}\cdot 3^{-1}(H_4 H_{24} - I_{28}) - 3^{-1} H_8^2 I_{12},\\
I_{28} &:= 2^{-1}\cdot 3^{-1}(F_{10}K_{18} - H_4 H_8 I_{16}),\\
H_{36} &:= 2^{-1}\cdot 3^{-2}(H_{12}H_{24} - I_{16}H_{20}) + 7\cdot 3^{-2} H_8 H_{28} + 3^{-1} H_8^3 H_{12},\\
I_{36} &:= K_{18}^2, \qquad J_{36} := E_6 K_{30},\\
H_{40} &:= 2^{-2}(H_4 H_{36} - 2^{-1}\cdot 3^{-1} F_{10} K_{30}) - 5\cdot 2^{-3}\cdot 3^{-1} H_4 H_8 H_{28},\\
&\qquad + 2^{-2} {H_8}^3 H_{16} + 2^{-1} H_8 I_{12} H_{20},\\
I_{40} &:= 2^{-1}\cdot 3^{-1}(F_{10}K_{30} - H_4 H_8 H_{28}),
\end{aligned}
$$

Construction of generators over **Z**

$$H_{48} := 2^{-2}(H_{12}H_{36} - H_{24}^2) - 2^{-3}H_8(H_{12}H_{28} + 2H_{40} + 4H_8H_{10}^2H_{12} - 2H_4H_8^2H_{20} - 2H_4H_8^3H_{12} + 4H_8I_{12}H_{20} + 2H_8^2H_{12}I_{12} - I_{16}H_{24} - 2H_8^3I_{16} + 2I_{40}),$$

$$I_{48} := K_{18}K_{30},$$

$$H_{52} := 2^{-1}\cdot 3^{-1}(F_{10}K_{42} - 2H_8F_{10}^2H_{12}^2 - 2^2H_8H_{12}I_{12}H_{20} - 5H_8F_{10}I_{12}K_{22} - H_8I_{16}H_{28} - H_8^3I_{12}I_{16}),$$

$$H_{60} := K_{30}^2, \qquad I_{60} := K_{18}K_{42}, \qquad H_{72} := K_{30}K_{42}, \qquad H_{84} := K_{42}^2.$$

Structure theorem over **Z** (main result)

Theorem (K):

(1) All Fourier coefficients of them are in $\mathbb{Z}$,

(2) $A^{(4)}(U_2(\mathcal{O});\mathbb{Z})$ is generated over $\mathbb{Z}$ by 24 modular forms

$$H_4,\ H_8,\ H_{12},\ I_{12},\ J_{12},\ H_{16},\ I_{16},\ H_{20},\ H_{24},\ I_{24},\ H_{28},\ I_{28},$$
$$H_{36},\ I_{36},\ J_{36},\ H_{40},\ I_{40},\ H_{48},\ I_{48},\ H_{52},\ H_{60},\ I_{60},\ H_{72},\ H_{84}.$$

(2) $\iff$ $\forall F \in M_k(U_2(\mathcal{O});\mathbb{Z})$ with $4 \mid k$, $\exists P \in \mathbb{Z}[\boldsymbol{X}]$ (24 variables) s.t.

$$F = P(H_4, H_8, H_{12}, \cdots, H_{84}).$$

§4. Proofs (without details)

Igusa's generators over **Z**

- $M_k(\Gamma_2;\mathbb{Z})$: Siegel modular forms of wt k for $\Gamma_2 := Sp_2(\mathbb{Z})$, s.t. FC are in $\mathbb{Z}$,

  Define
$$A^{(m)}(\Gamma_n;R) := \bigoplus_{k\in m\mathbb{Z}} M_k(\Gamma_2;R) \quad \text{(algebra } /R).$$

Theorem (Igusa (1979)):

$\exists X_k \in M_k(\Gamma_2;\mathbb{Z})$ $(k = 4, 6, 10, 12, 16, 18, 24, 28, 30, 36, 40, 42, 48)$,

$\exists Y_{12} \in M_{12}(\Gamma_2;\mathbb{Z})$ s.t.
$$A^{(2)}(\Gamma_2;\mathbb{Z}) = \mathbb{Z}[\underbrace{X_4, X_6, \cdots, X_{48}}_{\text{14 generators}}].$$

Variant of Igusa's result

Easy conclusion of Igusa's result:

$A^{(4)}(\Gamma_2;\mathbb{Z})$ is generated over $\mathbb{Z}$ by 23 generators;

$$
\begin{aligned}
&S_4 := X_4, \quad S_{12} := X_{12}, \quad T_{12} := Y_{12}, \quad U_{12} := X_6^2,\\
&S_{16} := X_6X_{10}, \quad T_{16} := X_{16}, \quad S_{20} := X_{10}^2, \quad S_{24} := X_{24},\\
&T_{24} := X_6X_{18}, \quad S_{28} := X_{28}, \quad T_{28} := X_{10}X_{18}, \quad S_{36} := X_{36},\\
&T_{36} := X_{18}^2, \quad U_{36} := X_6X_{30}, \quad S_{40} := X_{40}, \quad T_{40} := X_{10}X_{30},\\
&S_{48} := X_{48}, \quad T_{48} := X_{18}X_{30}, \quad S_{52} := X_{10}X_{42}, \quad S_{60} := X_{30}^2,\\
&T_{60} := X_{18}X_{42}, \quad S_{72} := X_{30}X_{42}, \quad S_{84} := X_{42}^2.
\end{aligned}
$$

Restriction map

- $\mathbb{S}_2 = \mathbb{H}_2 \cap \mathrm{Sym}_2(\mathbb{C}) \subset \mathbb{H}_2$: Siegel upper half sp.,
- We can define a restriction map $M_k(U_2(\mathcal{O})) \xrightarrow{|_{\mathbb{S}_2}} M_k(\Gamma_2)$,
- We have

$$a_{F|_{\mathbb{S}_2}}(m,r,n) = \sum_{\substack{s\in\mathbb{Z} \\ 4mn-(r^2+s^2)\geq 0}} a_F(m,r,s,n).$$

Here $(m,r,n) = \begin{pmatrix} m & \frac{r}{2} \\ \frac{r}{2} & n \end{pmatrix}$, $\quad (m,r,s,n) = \begin{pmatrix} m & \frac{r+si}{2} \\ \frac{r-si}{2} & n \end{pmatrix} \in \Lambda_2(\boldsymbol{K})$.

Restriction of our generators

We proved (in [Nagaoka-K])

$$H_4|_{\mathbb{S}_2} = X_4, \quad E_6|_{\mathbb{S}_2} = X_6, \quad H_8|_{\mathbb{S}_2} = 0, \quad F_{10}|_{\mathbb{S}_2} = 6X_{10}, \quad H_{12}|_{\mathbb{S}_2} = X_{12}.$$

This implies

Relation among generators:

$$H_{k_1}|_{\mathbb{S}_2} = S_{k_1}, \; I_{k_2}|_{\mathbb{S}_2} = T_{k_2}, \quad \text{and} \quad J_{k_3}|_{\mathbb{S}_2} = U_{k_3}$$

for each k_1, k_2, k_3 with

$$k_1 \in \{4, 12, 16, 20, 24, 28, 36, 40, 48, 52, 60, 72, 84\},$$
$$k_2 \in \{12, 16, 24, 28, 36, 40, 48, 60\}, \quad k_3 \in \{12, 36\}.$$

For example

$$I_{12} := 2^{-6} \cdot 3^{-3}(H_4^3 - E_6^2) + 2^4 \cdot 3^2 H_{12}$$

$$\longrightarrow \quad T_{12} := 2^{-6} \cdot 3^{-3}(X_4^3 - X_6^2) + 2^4 \cdot 3^2 X_{12},$$

$$H_{16} := 2^{-1} \cdot 3^{-1}(E_6 F_{10} - H_4^2 H_8) \quad \longrightarrow \quad S_{16} := X_6 X_{10},$$

$$I_{16} := 2^{-2} \cdot 3^{-1}(H_4 H_{12} - H_{16}) \longrightarrow T_{16} := 2^{-2} \cdot 3^{-1}(X_4 X_{12} - X_6 X_{10}),$$

$$H_{20} := 2^{-2} \cdot 3^{-2}(F_{10}^2 - H_4 H_8^2 - 2^2 \cdot 3 H_8 H_{12}) \quad \longrightarrow \quad S_{20} := X_{10}^2.$$

Proof of the structure theorem

First we prove

$$\text{All FC of our 24 generators are in } \mathbb{Z} \quad \Longrightarrow \quad \text{Structure theorem is true.}$$

Proof. Induction on the wt.

$$\dim_{\mathbb{C}} M_4(U_2(\mathcal{O})) = 1, \quad \dim_{\mathbb{C}} M_8(U_2(\mathcal{O})) = 2 \quad \Longrightarrow$$

$$M_4(U_2(\mathcal{O}); \mathbb{Z}) = H_4\mathbb{Z}, \quad M_8(U_2(\mathcal{O})_2; \mathbb{Z}) = H_4^2\mathbb{Z} \oplus H_8\mathbb{Z}.$$

Suppose that the statement is true for $\forall k < k_0$ with $4 \mid k$, $4 \mid k_0$.

$$F \in M_{k_0}(U_2(\mathcal{O}); \mathbb{Z}) \quad \Longrightarrow \quad F|_{\mathbb{S}_2} \in M_{k_0}(\Gamma_2; \mathbb{Z})$$

Proof of the structure theorem

$$\overset{\text{Igusa}}{\Longrightarrow} \quad \exists P \in \mathbb{Z}[\boldsymbol{X}] \text{ (23 variables) s.t. } F|_{\mathbb{S}_2} = P(S_4, S_{12}, T_{12}, \cdots, S_{84})$$

$$\Longrightarrow \quad G := F - P(H_4, H_{12}, I_{12}, \cdots, H_{84}) \in M_{k_0}(U_2(\mathcal{O}); \mathbb{Z})$$

$$\text{and} \qquad G|_{\mathbb{S}_2} = 0$$

$$\overset{\text{Dern-Krieg}}{\Longrightarrow} \quad \exists F' \in M_{k_0-8}(U_2(\mathcal{O}); \mathbb{Z}) \text{ s.t. } G = H_8 F'$$

$$\overset{\text{Induct. Hypo.}}{\Longrightarrow} \quad \exists Q \in \mathbb{Z}[\boldsymbol{X}] \text{ (24 variables) s.t. } F' = Q(H_4, H_8, H_{12}, \cdots, H_{84})$$

$$\Longrightarrow \quad F = P(H_4, H_{12}, I_{12}, \cdots, H_{84}) + H_8 Q(H_4, H_8, H_{12}, \cdots, H_{84})$$

Proof of integralities

We write

$$R[\![\boldsymbol{q}]\!] := \left\{ \sum_{H \in \Lambda_2(\boldsymbol{K})} a(H) e^{2\pi i \operatorname{tr}(HZ)} \;\middle|\; a(H) \in R \right\}.$$

$$H_4 := 1 + 2^4 \cdot 3S, \quad E_6 := 1 + 2^3 \cdot 3^2 T \qquad \text{with} \qquad S, T \in \mathbb{Z}[\![\boldsymbol{q}]\!]$$

Euler cong.

$$\Longrightarrow \quad S \equiv T \bmod 2^2 \cdot 3 \quad \Longrightarrow \quad \exists U \in \mathbb{Z}[\![\boldsymbol{q}]\!] \text{ s.t. } T = S + 2^2 \cdot 3U.$$

Key Fact 1.

$$H_4 = 1 + 2^4 \cdot 3S, \quad E_6 = 1 + 2^3 \cdot 3^2 S + 2^5 \cdot 3^3 U$$

Integrality of I_{16}

$I_{16} \cdots$ A maass lift of ell. MF for $\Gamma_0^{(1)}(4)$;

$$h_{15} := \theta^{14} f_2^4 - 28\theta^{10} f_2^5 + 192\theta^6 f_2^6$$
$$= q^4 + 12q^6 + 64q^7 + 36q^8 - 128q^{10} - 1152q^{11} - 936q^{12} - 504q^{14} \cdots$$
$$\left(\theta := 1 + 2\sum_{n\geq 1} q^{n^2}, \quad f_2 := \sum_{n:\ \mathrm{odd}} \sigma_1(n) q^n\right)$$

$$\Longrightarrow I_{16} = 2^{-2} \cdot 3^{-1}(H_4 H_{12} - H_{16}) \in \mathbb{Z}[\![\boldsymbol{q}]\!]$$

$$\Longrightarrow 6H_4H_{12} - E_6F_{10} + H_4^2H_8 = 2^3 \cdot 3^2 I_{16} \equiv 0 \bmod 2^3 \cdot 3^2$$

$$((\because)\ H_{16} = 2^{-1} \cdot 3^{-1}(E_6F_{10} - H_4^2H_8))$$

Integralities of all generators

$$\Longrightarrow 6H_{12} - F_{10} + H_4^2 H_8 \equiv 0 \mod 2^3 \cdot 3^2$$

$$((\because)\ H_4 \equiv 1 \mod 2^4 \cdot 3, \qquad E_6 \equiv 1 \mod 2^3 \cdot 3^2)$$

Key Fact 2.

$$\exists V \in \mathbb{Z}[\![\boldsymbol{q}]\!] \text{ s.t. } F_{10} = 6H_{12} + H_4^2 H_8 - 2^3 \cdot 3^2 V$$

By this relation, we can prove all generators are elements of

$$\mathbb{Z}[H_{12}, H_8, S, U, V]$$

An example of such the description

$$H_{20} = 2^{-2} \cdot 3^{-2}(F_{10}^2 - 12H_{12}H_8 - H_4H_8^2) \quad \text{(By Def)},$$

$$F_{10} = 6H_{12} + H_4^2H_8 - 2^3 \cdot 3^2V,$$
$$H_4 = 1 + 2^4 \cdot 3S, \quad E_6 = 1 + 2^3 \cdot 3^2S + 2^5 \cdot 3^3U.$$

We have

$$H_{20} = H_{12}^2 + 32H_{12}H_8S + 4H_8^2S + 768H_{12}H_8S^2 + 384H_8^2S^2 + 12288H_8^2S^3 + 147456H_8^2S^4 + 24H_{12}V + 4H_8V + 384H_8SV + 9216H_8S^2V + 144V^2$$

$$\Longrightarrow H_{20} \in M_{20}(U_2(\mathcal{O}); \mathbb{Z}).$$

§5. Problems

The remaining problems

(1) The structure of $A^{(2)}(U_2(\mathcal{O});\mathbb{Z})$,

(2) For the case $\boldsymbol{K} = \mathbb{Q}(\sqrt{-3})$.

Remarks on (1).

(1-1) The dificulties come from the factor 6 of $F_{10}|_{\mathbb{S}_2} = 6X_{10}$.

(1-2) $\exists K_{46} \in M_{46}(U_2(\mathcal{O});\mathbb{Z})$, $\exists K_{58} \in M_{58}(U_2(\mathcal{O});\mathbb{Z})$ s.t.

$$K_{46}|_{\mathbb{S}_2} = X_{10}X_{36}, \qquad K_{58}|_{\mathbb{S}_2} = X_{10}X_{48}$$

only I think!

$\Longrightarrow$ we can obtain (1). … I don't believe the existences

Eisenstein field case

$$\boldsymbol{K} := \mathbb{Q}(\sqrt{-3}) \overset{\text{Nagaoka-K}}{\Longrightarrow}$$

$$A^{(2)}(U_2(\mathcal{O}); \mathbb{Z}[1/2, 1/3]) = \mathbb{Z}[1/2, 1/3][E_4, E_6, F_{10}, F_{12}, \chi_{18}]$$

and

$$E_4|_{\mathbb{S}_2} = X_4, \quad E_6|_{\mathbb{S}_2} = X_6, \quad F_{10}|_{\mathbb{S}_2} = 2X_{10}, \quad F_{12}|_{\mathbb{S}_2} = 2X_{12}, \quad \chi_{18}|_{\mathbb{S}_2} = 0.$$

Thank you for your attention!